MODIFIED REGULAR REPRESENTATIONS OF AFFINE AND VIRASORO ALGEBRAS, VOA STRUCTURE AND SEMI-INFINITE COHOMOLOGY.

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Abstract. We find a counterpart of the classical fact that the regular representation $\mathfrak{R}(G)$ of a simple complex group G is spanned by the matrix elements of all irreducible representations of G. Namely, the algebra of functions on the big cell $G_0 \subset G$ of the Bruhat decomposition is spanned by matrix elements of big projective modules from the category $\mathcal{O}$ of representations of the Lie algebra $\mathfrak{g}$ of G, and has the structure of a $\mathfrak{g} \oplus \mathfrak{g}$-module.

We extend both regular representations to the affine group $\hat{G}$, and show that the loop form of the Bruhat decomposition of $\hat{G}$ yields modified versions of $\mathfrak{R}(\hat{G})$. They involve pairings of positive and negative level modules, with the total value of the central charge required for the existence of non-trivial semi-infinite cohomology. In this paper we consider in detail the case $G = SL(2, \mathbb{C})$, the corresponding finite-dimensional and affine Lie algebras, and the closely related to them Virasoro algebra.

Using the Fock space realization, we show that both types of modified regular representations for the affine and Virasoro algebras become vertex operator algebras, whereas the ordinary regular representations have instead the structure of conformal field theories. We identify the inherited algebra structure on the semi-infinite cohomology when the central charge is generic. We conjecture that for the integral values of the central charge the semi-infinite cohomology coincides with the Verlinde algebra and its counterpart associated with the big projective modules.

0. Introduction.

The study of the regular representation of a simple complex Lie group G is at the foundation of representation theory of G. Realized as the space of regular functions on G, the regular representation $\mathfrak{R}(G)$ carries two compatible structures of a G-bimodule and of a commutative associative algebra. An algebro-geometric version of the Peter-Weyl theorem establishes the decomposition of $\mathfrak{R}(G)$ into a direct sum of subspaces, spanned by matrix elements of all irreducible finite-dimensional representations V_λ of G, indexed by integral dominant highest weights $\lambda \in \mathbf{P}^+$. In other words, we have an isomorphism of G-bimodules

$$\mathfrak{R}(G) = \bigoplus_{\lambda \in \mathbf{P}^+} V_\lambda \otimes V_\lambda^*. \tag{0.1}$$

where V_λ^* is the dual representation of G. The multiplication in $\mathfrak{R}(G)$ can be described in representation-theoretic terms as a pairing of intertwining operators for the left and right $\mathfrak{g}$-actions with appropriate structural coefficients. Thus the algebra structure on $\mathfrak{R}(G)$ encodes the information about the tensor category of finite-dimensional $\mathfrak{g}$-modules.

The representations of G can also be viewed as modules over the simple complex Lie algebra $\mathfrak{g}$ associated with G. In the case of the Lie algebra $\mathfrak{g}$ it is natural to consider a larger collection of modules - namely, the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Infinite-dimensional $\mathfrak{g}$-modules from the category $\mathcal{O}$ are not integrable, and therefore their matrix

elements cannot be regarded as functions on G. However, one can interpret them as functions on the open dense subset $G^o \subset G$, given by the Gauss decomposition

$$G^o = N_- \cdot T \cdot N_+, \tag{0.2}$$

where $N_\pm$ is the upper and lower triangular unipotent subgroup of G, and T is the diagonal maximal abelian subgroup. The space $\mathfrak{R}(G^o)$ of regular functions on G^o does not have the structure of a representation of G. Nevertheless, the left and right infinitesimal actions of the Lie algebra $\mathfrak{g}$ on this space are well-defined, and can be expressed in terms of explicit differential operators in the parameters of the Gauss decomposition (0.2). The enlarged regular representation $\mathfrak{R}(G^o)$ decomposes into the direct sum of bimodules spanned by the matrix coefficients of all "big" projective $\mathfrak{g}$-modules P_λ, indexed by strictly anti-dominant highest weights $\lambda \in -\mathbf{P}^{++} = -(\mathbf{P}^+ + \rho)$, where ρ is the half-sum of all positive roots of $\mathfrak{g}$. Thus we obtain an isomorphism of $\mathfrak{g}$-bimodules

$$\mathfrak{R}(G^o) \cong \bigoplus_{\lambda \in -\mathbf{P}^{++}} (P_\lambda \otimes P_\lambda^*) \Big/ I_\lambda, \tag{0.3}$$

where P_λ^* is the module dual to P_λ, and I_λ is the sub-bimodule of the matrix coefficients which vanish identically on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

It is important to note that the dual modules P_λ^* do not belong to the category $\mathcal{O}$, but to its "mirror image", in which all highest weight modules are replaces by lowest weight modules. In order to stay in the category $\mathcal{O}$ we replace the open subset G^o coming from the Gauss decomposition by the maximal cell in the Bruhat decomposition

$$G_0 = N_+ \cdot \mathbf{w_0} \cdot T \cdot N_+, \tag{0.4}$$

where $\mathbf{w_0}$ is the longest element of the Weyl group W. Then we obtain a version of the isomorphism (0.3),

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda \in -\mathbf{P}^{++}} (P_\lambda \otimes P_\lambda^\star) \Big/ I_\lambda, \tag{0.5}$$

where the 'twisted' duals $P_\lambda^\star$ differs from P_λ^* by the automorphism ω of $\mathfrak{g}$, which is induced by $\mathbf{w}_0$ and interchanges the positive and negative roots.

The theorems of Peter-Weyl type and the Gauss decomposition can be extended to the central extension of the loop group $\hat{G}$ associated to G, and to the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ and its universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}})$. In this infinite-dimensional case the space $\mathfrak{R}(\hat{G})$ of regular functions on $\hat{G}$ is decomposed into the direct sum of subspaces $\mathfrak{R}_k(\hat{G})$, corresponding to the value $k \in \mathbb{Z}$ of the central charge. Using the version of the Gauss decomposition (0.2) known as the Birkhoff decomposition, one can show (see [PS]) that for any $k \in \mathbb{Z}$ there is an isomorphism

$$\mathfrak{R}_k(\hat{G}) \cong \bigoplus_{\lambda \in \mathbf{P}_+^k} \hat{V}_{\lambda,k} \otimes \hat{V}_{\lambda,k}^*, \tag{0.6}$$

where λ runs over the truncated alcove $\mathbf{P}_k^+ \subset \mathbf{P}^+$, depending on k, and $\hat{V}_{\lambda,k}$ are the corresponding irreducible modules. Similarly, one can obtain decompositions of $\mathfrak{R}_k(\hat{G}^o)$ analogous to (0.3), where $\hat{G}^o$ is the maximal cell in the Birkhoff decomposition. Viewing the decomposition (0.6) in terms of the Lie algebra $\hat{\mathfrak{g}}$ allows to extend it for all values of k, with $\mathbf{P}_k^+ = \mathbf{P}^+$ for $k \notin \mathbb{Q}$.

To transform the dual module $\hat{V}^*_{\lambda,k}$ into a module from the category $\mathcal{O}$ for $\hat{\mathfrak{g}}$, one might apply again an automorphism of $\hat{\mathfrak{g}}$ which interchanges the positive and negative affine roots. However, it no longer belongs to the affine Weyl group, and the Bruhat decomposition for $\hat{G}$ does not have a maximal cell. To overcome this obstacle we consider instead an intermediate between the Birkhoff and the affine Bruhat decompositions - the loop version of the finite-dimensional Bruhat decomposition, and the corresponding big cell

$$\hat{G}_0 = LN_+ \cdot \mathbf{w_0} \cdot \widehat{LT} \cdot LN_+, \tag{0.7}$$

where $LN_\pm$ denote the loop groups with values in $\mathfrak{n}_\pm$, and $\widehat{LT}$ is the central extension of the loop group with values in T. The decomposition (0.7) is especially useful for explicit realizations of the left and right regular $\hat{\mathfrak{g}}$-actions in terms of differential operators. However, we are still in "semi-infinite" distance from the category $\mathcal{O}$, and need to further apply a well-known procedure of "changing the vacuum", which has originated from the free field realizations of the Wakimoto modules and the irreducible representations $\hat{V}_{\lambda,k}$ (see [FeFr1, BF]). As a result of this procedure we obtain a modified affine version of the extended regular representation (0.5),

$$\mathfrak{R}'_k(\hat{G}_0) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} \left(\hat{P}_{\lambda,k-h^\vee} \otimes \hat{P}^\star_{\lambda,-k-h^\vee}\right) \Big/ \hat{I}_{\lambda,k}, \tag{0.8}$$

where $\hat{P}_{\lambda,k-h^\vee}$ and $\hat{P}^\star_{\lambda,-k-h^\vee}$ are the projective $\hat{\mathfrak{g}}$-modules and their 'twisted' duals, $\hat{I}_{\lambda,k}$ are appropriate sub-bimodules, and we assume $k \notin \mathbb{Q}$. The levels are shifted by the dual Coxeter number $h^\vee$, so that the diagonal $\hat{\mathfrak{g}}$-action has the level $-2h^\vee$.

Like the Wakimoto modules, the bimodule $\mathfrak{R}'_k(\hat{G}_0)$ is realized as a certain Fock space, with two commuting $\hat{\mathfrak{g}}$-actions described explicitly. This realization is similar to the standard realization of the Wakimoto modules, but the actions of $\hat{\mathfrak{g}}$ contain a crucial new ingredient - the vertex operators, directly related to the screening operators used to construct intertwining operators for the affine Lie algebra. We also establish that for $k \notin \mathbb{Q}$ the structure of the socle filtration of the non-semisimple bimodule $\mathfrak{R}'_k(\hat{G}_0)$ is the same as in the finite-dimensional case. In particular, $\mathfrak{R}'_k(\hat{G}_0)$ contains the distinguished sub-bimodule

$$\mathfrak{R}'_k(\hat{G}) \cong \bigoplus_{\lambda\in\mathbf{P}^+} \hat{V}_{\lambda,k-h^\vee} \otimes \hat{V}^\star_{\lambda,-k-h^\vee}. \tag{0.9}$$

The shifts of the central charge by the dual Coxeter number no longer allow the interpretation of the bimodules in (0.8) and (0.9) as spaces of matrix elements of $\hat{\mathfrak{g}}$-modules. Nevertheless, the structures of these bimodules are completely analogous to those of bimodules $\mathfrak{R}(G_0)$ and $\mathfrak{R}(G)$!

The vacuum module $\hat{V}_{0,k}$ of the affine Lie algebra $\hat{\mathfrak{g}}$ carries an extremely rich additional structure of a vertex operator algebra (VOA); other $\hat{\mathfrak{g}}$-modules $\hat{V}_{\lambda,k}$ become its representations (see [FZ]). We show in this paper that the bimodules $\mathfrak{R}'_k(\hat{G})$ and $\mathfrak{R}'_k(\hat{G}_0)$ also admit a vertex operator structure, compatible with $\hat{\mathfrak{g}}$-actions. In the proof we use the explicit Fock space realization of these bimodules. As in the finite-dimensional case, the VOA structure of the modified regular representations $\mathfrak{R}'_k(\hat{G})$ and $\mathfrak{R}'_k(\hat{G}_0)$ encodes the information about fusion rules of the corresponding tensor categories of $\hat{\mathfrak{g}}$-modules. Thus besides the vacuum modules there is a class of vertex operator algebras associated to affine Lie algebras with fixed central charges. It is also related to the algebras of chiral differential operators over

the simple algebraic group G recently studied in [GMS] and [AG]. A study of this relation might help to understand the geometric nature of the modified regular representations.

On the other hand, the original regular representation $\mathfrak{R}_k(\hat{G})$ does not seem to have a VOA structure. Instead it has the structure of a two-dimensional conformal field theory, which is an object of a different nature despite having local properties similar to those of a VOA. The bimodule $\mathfrak{R}_k(\hat{G}_0)$ has a structure of a generalized (non-semisimple!) conformal field theory.

It is well-known that the representation theory of $\hat{\mathfrak{g}}$ is closely related to the representation theory of the corresponding $\mathcal{W}$-algebra via the quantum Drinfeld-Sokolov reduction. In particular, one expects to have an analogue of the Peter-Weyl theorem for the $\mathcal{W}$-algebras. In this paper we consider in detail the simplest case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, when the corresponding $\mathcal{W}$-algebra is the infinite-dimensional Virasoro algebra. We give explicit realizations of the Virasoro bimodules, analogous to (0.8) and (0.9), and equip them with compatible VOA structures. The structures of the non-semisimple modified regular representations are quite parallel in all cases; we fully describe their socle filtrations. Generalizations of our constructions to higher rank Lie algebras are straightforward, but their $\mathcal{W}$-algebra versions require more technicalities; full details will be presented in a subsequent paper.

A remarkable feature of all the modified bimodules that appear in the decompositions of Peter-Weyl type is that the central charge of the diagonal subalgebra is always equal to the special values that appear in the semi-infinite cohomology theory [Fe, FGZ] - namely, $-2h^\vee$ for the affine Lie algebras and 26 for Virasoro. Moreover, thanks to a general result of [LZ1], the corresponding semi-infinite cohomology spaces inherit a VOA structure from the modified regular representations. In our case, they degenerate into commutative associative superalgebras, and for generic central charge we establish isomorphisms between cohomology groups with coefficients in the corresponding modified regular representations of the affine and Virasoro algebras and their finite-dimensional counterparts. In particular, we show that the 0th semi-infinite cohomologies of the affine and Virasoro algebras are isomorphic to the Grothendieck ring of finite-dimensional representations of G. We conjecture that for integral k they lead to the Verlinde algebra and its projective counterpart.

This paper is organized as follows. In Section 1 we consider the Bruhat decomposition and the Peter-Weyl theorems in the finite-dimensional case with $G = SL(2, \mathbb{C})$. We give a Fock space realization of the algebra $\mathfrak{R}(G_0)$, and obtain explicit formulas for the $\mathfrak{g}$-actions and decomposition theorems, which will later be used as prototypes of the infinite-dimensional case. In the last subsection we compute the Lie algebra cohomology with coefficients in $\mathfrak{R}(G)$ and $\mathfrak{R}(G_0)$. In Section 2 we study the affine case, and use the loop version of the finite-dimensional Bruhat decomposition to obtain the modified Peter-Weyl theorems for the spaces $\mathfrak{R}'_k(\hat{G})$ and $\mathfrak{R}'_k(\hat{G}_0)$. The Fock space realization of these spaces equips them with VOA structures compatible with the regular $\hat{\mathfrak{g}}$-actions. The semi-infinite cohomology of $\hat{\mathfrak{g}}$ with coefficients in the modified regular representations $\mathfrak{R}'_k(\hat{G})$ and $\mathfrak{R}'_k(\hat{G}_0)$ for generic central charge is shown to be isomorphic to its finite-dimensional counterpart. In Section 3 we construct the analogues of the modified regular representations of the Virasoro algebra using the quantum Drinfeld-Sokolov reduction. We compute the corresponding semi-infinite cohomology groups using methods developed in string theory, and prove that they are isomorphic to their affine counterparts. Finally, in Section 4 we describe another class of vertex operator algebras obtained by the pairing of $\widehat{\mathfrak{sl}}(2, \mathbb{C})$ and Virasoro modules. We also discuss generalizations of our results to Lie algebras of other types, and to the integral values of the

central charge. We conclude with conjectures on relations of the semi-infinite cohomology of $\mathfrak{R}'_k(\hat{G})$ and $\mathfrak{R}'_k(\hat{G}_0)$ for $k \in \mathbb{Z}_{>0}$ with the Verlinde algebra, its projective counterpart and twisted equivariant K-theory.

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1. Regular representation of $\mathfrak{sl}(2,\mathbb{C})$ on the big cell.

1.1. **Regular representations of $\mathfrak{sl}(2,\mathbb{C})$.** Let $G = SL(2,\mathbb{C})$. We define the left and right regular actions of G on the space $\mathfrak{R}(G)$ of regular functions on G by

$$(\pi_l(g)\psi)(h) = \psi(g^{-1}h), \qquad (\pi_r(g)\psi)(h) = \psi(h\,g), \qquad g,h \in G. \tag{1.1}$$

The multiplication in $\mathfrak{R}(G)$ intertwines both left and right regular actions.

Let T, N_+ denote the diagonal and unipotent upper-triangular subgroups of G. The group $W = \mathrm{Norm}(T)/T$ is called the Weyl group. The Bruhat decomposition $G = N_+ \cdot W \cdot T \cdot N_+$ implies that every $g \in G$ can be factored as $g = n \cdot w \cdot t \cdot n'$ for some $n, n' \in N_+, t \in T, w \in W$.

We denote by G_0 the big cell of the Bruhat decomposition, corresponding to the longest Weyl group element $\mathbf{w}_0$. Explicitly, G_0 is the dense open subset of G, consisting of $g \in G$,

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

for some $x, y \in \mathbb{C}$ and $\zeta \in \mathbb{C}^\times$. The variables x, y, ζ can be viewed as coordinates on G_0, and thus the algebra $\mathfrak{R}(G_0)$ of regular functions on G_0 is identified with the space $\mathbb{C}[x, y, \zeta^{\pm 1}]$.

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ be the Lie algebra of G, with the standard basis

$$\mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying the commutation relations $[\mathbf{h},\mathbf{e}] = 2\mathbf{e},\ [\mathbf{h},\mathbf{f}] = -2\mathbf{f},\ [\mathbf{e},\mathbf{f}] = \mathbf{h}$. The nilpotent subalgebras $\mathfrak{n}_\pm$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are defined by $\mathfrak{n}_+ = \mathbb{C}\mathbf{e}$, $\mathfrak{h} = \mathbb{C}\mathbf{h}$, $\mathfrak{n}_- = \mathbb{C}\mathbf{f}$. The element $\mathbf{w_0} \in W$ determines a Lie algebra involution ω of $\mathfrak{g}$, such that $\omega(\mathfrak{n}_\pm) = \mathfrak{n}_\mp$ and $\omega(\mathfrak{h}) = \mathfrak{h}$, defined by

$$\omega(\mathbf{e}) = -\mathbf{f}, \qquad \omega(\mathbf{h}) = -\mathbf{h}, \qquad \omega(\mathbf{f}) = -\mathbf{e}. \tag{1.3}$$

The infinitesimal regular actions of $\mathfrak{g}$ on $\mathfrak{R}(G)$, corresponding to (1.1), are given by

$$(\pi_l(x)\psi)(g) = \frac{d}{dt}\psi(e^{-t\,x}g)\Big|_{t=0}, \qquad (\pi_r(x)\psi)(g) = \frac{d}{dt}\psi(g\,e^{t\,x})\Big|_{t=0}, \qquad x \in \mathfrak{g},\ g \in G. \tag{1.4}$$

These formulas also define left and right infinitesimal actions of $\mathfrak{g}$ on the space $\mathfrak{R}(G_0)$. (These actions cannot be lifted to the group G, because G_0 is not invariant under left and right shifts). Elementary calculations yield the following explicit description of the regular $\mathfrak{g}$-actions (cf. [FeP]).

Proposition 1.1. *The regular $\mathfrak{g}$-actions on $\mathfrak{R}(G_0)$ are given by*

$$\begin{aligned} \pi_l(\mathbf{e}) &= -\partial_x, \\ \pi_l(\mathbf{h}) &= \zeta\partial_\zeta - 2x\partial_x, \\ \pi_l(\mathbf{f}) &= -x\zeta\partial_\zeta + x^2\partial_x + \zeta^{-2}\partial_y. \end{aligned} \tag{1.5}$$

$$\begin{aligned} \pi_r(\mathbf{e}) &= \partial_y, \\ \pi_r(\mathbf{h}) &= \zeta\partial_\zeta - 2y\partial_y, \\ \pi_r(\mathbf{f}) &= y\zeta\partial_\zeta - y^2\partial_y - \zeta^{-2}\partial_x. \end{aligned} \tag{1.6}$$

1.2. **Bosonic realizations.** We now reformulate the constructions of the previous section in terms of Fock modules for certain Heisenberg algebras. These realizations admit generalizations to the affine and Virasoro cases, where the geometric approach to the regular representations becomes more subtle.

The operators $\beta = x,\ \gamma = -\partial_x$ acting on polynomials in y give a representation of the Heisenberg algebra with generators β, γ and relation $[\beta, \gamma] = 1$. The polynomial space $\mathbb{C}[y]$ is then identified with its irreducible representation $F(\beta, \gamma)$, generated by a vector $\mathbf{1}$ satisfying $\gamma\,\mathbf{1} = 0$.

The operators $\bar\beta = -y,\ \bar\gamma = \partial_y$ generate a second Heisenberg algebra, acting irreducibly in the space $F(\bar\beta, \bar\gamma) \cong \mathbb{C}[x]$. Here and everywhere else in this paper the 'bar' notation is used to denote the second copies of algebras and their generators; it does not denote the complex conjugation.

We identify $\mathfrak{h}^* \cong \mathbb{C}$ so that $\mathbf{P} \cong \mathbb{Z}$. Whenever possible, we use the more invariant notation in order to avoid possible numeric coincidences. The operators $\mathbf{1}_\lambda = \zeta^\lambda$ and $a = \zeta\partial_\zeta$ gives rise to the semi-direct product $\mathbb{C}[a] \ltimes \mathbb{C}[\mathbf{P}]$, with $\mathbb{C}[a]$ acting on $\mathbb{C}[\mathbf{P}]$ by derivations: $a\mathbf{1}_\lambda = \lambda\mathbf{1}_\lambda$.

Thus, we get a realization of the algebra $\mathfrak{R}(G_0)$ of regular functions on G_0, with the $\mathfrak{g} \oplus \mathfrak{g}$-action described by the abstract versions of the formulas (1.5), (1.6).

Theorem 1.2. *The space $\mathbb{F} = F(\beta, \gamma) \otimes F(\bar\beta, \bar\gamma) \otimes \mathbb{C}[\mathbf{P}]$ gives a realization of the algebra $\mathfrak{R}(G_0)$. In particular,*

(1) *The space $\mathbb{F}$ has a $\mathfrak{g} \oplus \mathfrak{g}$-module structure, given by*

$$\begin{aligned} \mathbf{e} &= \gamma, \\ \mathbf{h} &= 2\,\beta\gamma + a, \\ \mathbf{f} &= -\beta^2\gamma - \beta\,a + \bar\gamma\,\mathbf{1}_{-2}, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \bar{\mathbf{e}} &= \bar\gamma, \\ \bar{\mathbf{h}} &= 2\,\bar\beta\bar\gamma + a, \\ \bar{\mathbf{f}} &= -\bar\beta^2\bar\gamma - \bar\beta\,a + \gamma\,\mathbf{1}_{-2}. \end{aligned} \tag{1.8}$$

(2) *The space $\mathbb{F}$ has a compatible commutative algebra structure (i.e. the multiplication in $\mathbb{F}$ intertwines the $\mathfrak{g} \oplus \mathfrak{g}$-action).*

By specializing the action (1.7) to the subspace $\ker\bar\gamma \subset \mathbb{F}$, we get the following well-known realizations of $\mathfrak{g}$-action in the spaces $F_\lambda = F(\beta, \gamma) \otimes \mathbb{C}\mathbf{1}_\lambda$:

$$\begin{aligned} \mathbf{e} &= \gamma, \\ \mathbf{h} &= 2\beta\gamma + \lambda, \\ \mathbf{f} &= -\beta^2\gamma - \lambda\,\beta. \end{aligned} \tag{1.9}$$

Remark 1. Simultaneous rescaling of the extra terms in (1.7),(1.8), involving the shift $\mathbf{1}_{-2}$, by any multiple ϵ would preserve all the $\mathfrak{g} \oplus \mathfrak{g}$ commutation relations. For $\epsilon = 0$ such $\mathfrak{g} \oplus \mathfrak{g}$-action degenerates into the product of two standard $\mathfrak{g}$-actions (1.9). However, the multiplication in this naïve bimodule loses much of its rich structure, and no longer encodes the information about the fusion rules in the tensor category of finite-dimensional $\mathfrak{g}$-modules.

1.3. **$\mathfrak{g}\oplus\mathfrak{g}$-module structure of the modified regular representation.** In this subsection we describe the socle filtration of the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{F}$.

For any $\lambda\in\mathfrak{h}^*$, we denote by V_λ the irreducible $\mathfrak{g}$-module, generated by a highest weight vector v_λ satisfying $\mathbf{e}\,v_\lambda=0$ and $\mathbf{h}\,v_\lambda=\lambda\,v_\lambda$.

Recall that a $\mathfrak{g}$-module V is said to have a weight space decomposition, if

$$V=\bigoplus_{\mu\in\mathfrak{h}^*}V[\mu],\qquad V[\mu]=\left\{v\in V\,\middle|\,\mathbf{h}\,v=\mu\,v\right\}.$$

The restricted dual space $V'=\bigoplus_{\mu\in\mathfrak{h}^*}V[\mu]'$ can be equipped with a $\mathfrak{g}$-action, defined by

$$\langle g\,v',v\rangle=-\langle v',\,\omega(g)\,v\rangle,\qquad g\in\mathfrak{g},\,v\in V,\,v'\in V',$$

where ω is as in (1.3). We denote the resulting dual module $V^\star$.

We have an involution $\lambda\mapsto\lambda^\star$ of $\mathfrak{h}^*$, determined by the condition $(V_\lambda)^\star\cong V_{\lambda^\star}$. This involution can also be defined by $\lambda^\star=-\mathbf{w_0}(\lambda)$, where $\mathbf{w_0}$ is the longest Weyl group element.

For $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$, we have $\lambda^\star=\lambda$. However, we keep the notation $\lambda^\star$, to indicate how our constructions generalize to Lie algebras of higher rank, where the involution is nontrivial.

Theorem 1.3. *There exists a filtration*

$$0\subset\mathbb{F}^{(0)}\subset\mathbb{F}^{(1)}\subset\mathbb{F}^{(2)}=\mathbb{F}\tag{1.10}$$

of $\mathfrak{g}\oplus\mathfrak{g}$-submodules of $\mathbb{F}$, such that

$$\mathbb{F}^{(2)}/\mathbb{F}^{(1)}\cong\bigoplus_{\lambda\in\mathbf{P}^+}V_{-\lambda-2}\otimes V^\star_{-\lambda-2},\tag{1.11}$$

$$\mathbb{F}^{(1)}/\mathbb{F}^{(0)}\cong\bigoplus_{\lambda\in\mathbf{P}^+}\left(V_\lambda\otimes V^\star_{-\lambda-2}\oplus V_{-\lambda-2}\otimes V^\star_\lambda\right),\tag{1.12}$$

$$\mathbb{F}^{(0)}\cong\bigoplus_{\lambda\in\mathbf{P}}V_\lambda\otimes V^\star_\lambda.\tag{1.13}$$

Proof. We introduce a filtration of $\mathfrak{g}\oplus\mathfrak{g}$-submodules of $\mathbb{F}$

$$\cdots\subset\mathbb{F}_{\le-2}\subset\mathbb{F}_{\le-1}\subset\mathbb{F}_{\le0}\subset\mathbb{F}_{\le1}\subset\mathbb{F}_{\le2}\subset\ldots,\tag{1.14}$$

satisfying $\bigcap_{\lambda\in\mathbf{P}}\mathbb{F}_{\le\lambda}=0$ and $\bigcup_{\lambda\in\mathbf{P}}\mathbb{F}_{\le\lambda}=\mathbb{F}$, where

$$\mathbb{F}_{\le\lambda}=F(\beta,\gamma)\otimes F(\bar\beta,\bar\gamma)\otimes\bigoplus_{\mu\le\lambda}\mathbb{C}\mathbf{1}_\mu,\qquad\lambda\in\mathbf{P}.$$

It is clear that $\mathbb{F}_{\le\lambda}/\mathbb{F}_{<\lambda}\cong F_\lambda\otimes F_{\lambda^\star}$; moreover, for $\lambda<0$ we have $F_\lambda\cong V_\lambda$, and for $\lambda\ge0$ there is a short exact sequence $0\to V_\lambda\to F_\lambda\to V_{-\lambda-2}\to0$. The linking principle for $\mathfrak{g}$-modules implies that the successive quotients $\mathbb{F}_{\le\lambda}/\mathbb{F}_{<\lambda}$ and $\mathbb{F}_{\le\mu}/\mathbb{F}_{<\mu}$ of this filtration may be non-trivially linked only if $\mu=-\lambda-2$.

Thus we see that the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{F}$ splits into the direct sum of blocks

$$\mathbb{F}=\mathbb{F}(-1)\oplus\bigoplus_{\lambda\in\mathbf{P}^+}\mathbb{F}(\lambda),\tag{1.15}$$

where $\mathbb{F}(-1)\cong V_{-1}\otimes V^\star_{-1}$, and $\mathbb{F}(\lambda)\cong\left(V_{-\lambda-2}\otimes V^\star_{-\lambda-2}\right)+(F_\lambda\otimes F_{\lambda^\star})$ for $\lambda\in\mathbf{P}^+$; another way to obtain the decomposition (1.15) is by using the Casimir operator.

It remains to describe the structure of $\mathbb{F}(\lambda)$ for each $\lambda \in \mathbf{P}^+$. By construction, $\mathbb{F}(\lambda)$ can be included in a short exact sequence $0 \to V_{-\lambda-2} \otimes V^\star_{-\lambda-2} \to \mathbb{F}(\lambda) \to F_\lambda \otimes F_{\lambda^\star} \to 0$. We conclude that there exists a filtration $0 \subset \mathbb{F}(\lambda)^{(0)} \subset \mathbb{F}(\lambda)^{(1)} \subset \mathbb{F}(\lambda)^{(2)} = \mathbb{F}(\lambda)$, such that

$$\begin{aligned} \mathbb{F}(\lambda)^{(2)}/\mathbb{F}(\lambda)^{(1)} &\cong V_{-\lambda-2} \otimes V^\star_{-\lambda-2}, \\ \mathbb{F}(\lambda)^{(1)}/\mathbb{F}(\lambda)^{(0)} &\cong \left(V_\lambda \otimes V^\star_{-\lambda-2}\right) \oplus \left(V_{-\lambda-2} \otimes V^\star_\lambda\right), \\ \mathbb{F}(\lambda)^{(0)} &\cong \left(V_{-\lambda-2} \otimes V^\star_{-\lambda-2}\right) + \left(V_\lambda \otimes V^\star_\lambda\right). \end{aligned}$$

In fact, the linking principle implies that the sum in $\mathbb{F}(\lambda)^{(0)}$ is direct:

$$\mathbb{F}(\lambda)^{(0)} \cong \left(V_{-\lambda-2} \otimes V^\star_{-\lambda-2}\right) \oplus \left(V_\lambda \otimes V^\star_\lambda\right).$$

Finally, we construct the filtration (1.10) by setting

$$\mathbb{F}^{(0)} = \mathbb{F}(-1) \oplus \bigoplus_{\lambda \in \mathbf{P}^+} \mathbb{F}(\lambda)^{(0)}, \qquad \mathbb{F}^{(1)} = \mathbb{F}(-1) \oplus \bigoplus_{\lambda \in \mathbf{P}^+} \mathbb{F}(\lambda)^{(1)},$$

which obviously satisfies the required conditions (1.11), (1.12), (1.13). □

Remark 2. For a Lie algebra $\mathfrak{g}$ of higher rank, we will get a similar filtration of length $2\,l(\mathbf{w_0}) + 1$, and in addition to the regular blocks, corresponding to $\lambda \in \mathbf{P}^+$, and the most degenerate block $\mathbb{F}(-1)$, there will be all intermediate types.

The natural inclusion of algebras $\mathfrak{R}(G) \subset \mathfrak{R}(G_0)$ can be seen in the Fock space realizations.

Corollary 1.4. *There exists a subspace* $\mathbf{F} \subset \mathbb{F}$ *satisfying the following properties.*

(1) $\mathbf{F}$ *is a subalgebra of* $\mathbb{F}$*, and is generated by the elements from the submodule* $V_1 \otimes V_1^\star$*, corresponding to the matrix elements of the canonical representation of* G.

(2) $\mathbf{F}$ *is a* $\mathfrak{g} \oplus \mathfrak{g}$*-submodule of* $\mathbb{F}$*, and is generated by the vectors* $\{\mathbf{1}_\lambda\}_{\lambda \in \mathbf{P}^+}$. *We have*

$$\mathbf{F} = \bigoplus_{\lambda \in \mathbf{P}^+} \mathbf{F}(\lambda) \cong \bigoplus_{\lambda \in \mathbf{P}^+} V_\lambda \otimes V^\star_\lambda. \tag{1.16}$$

(3) *The space* $\mathbf{F}$ *is a realization of the algebra* $\mathfrak{R}(G)$.

In the polynomial realization, the generators of $\mathbf{F}$ from $V_1 \otimes V_1^\star$ are identified with functions

$$\psi_{11} = \zeta, \qquad \psi_{12} = x\zeta, \qquad \psi_{21} = y\zeta, \qquad \psi_{22} = xy\zeta + \zeta^{-1},$$

which satisfy the relation $\psi_{11}\psi_{22} - \psi_{12}\psi_{21} = 1$. This establishes a very direct connection with the space of regular functions on the group $G = SL(2, \mathbb{C})$.

1.4. **The generalized Peter-Weyl theorem.** In this section we interpret the space $\mathfrak{R}(G_0)$ of regular functions on G_0 and its Fock space realization $\mathbb{F}$ as the algebra of matrix elements of all modules from the category $\mathcal{O}$.

Recall that the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ consists of all finitely generated, locally $\mathfrak{n}_+$-nilpotent $\mathfrak{g}$-modules. In particular, $V_\lambda \in \mathcal{O}$ for any λ. If $V \in \mathcal{O}$, then $V^\star \in \mathcal{O}$.

For any $\mathfrak{g}$-module V we define $\mathbb{M}(V)$ to be the subspace of $\mathcal{U}(\mathfrak{g})'$, spanned by functionals

$$\phi_{v,v'}(x) = \langle v', x\, v\rangle, \qquad v \in V,\ v' \in V',\ x \in \mathcal{U}(\mathfrak{g}), \tag{1.17}$$

where $\langle \cdot, \cdot \rangle$ stands for the natural pairing between V and V'. The functionals (1.17) are called matrix elements of the representation V.

Proposition 1.5. *Introduce a $\mathfrak{g}\oplus\mathfrak{g}$-module structure on the restricted dual $\mathcal{U}(\mathfrak{g})'$ by*

$$(\pi_l(g)\phi)(x) = \phi(xg), \qquad (\pi_r(g)\phi)(x) = -\phi(\omega(g)x) \tag{1.18}$$

for any $\phi\in\mathcal{U}(\mathfrak{g})'$, $g\in\mathfrak{g}$, $x\in\mathcal{U}(\mathfrak{g})$. Then

(1) *For any $\mathfrak{g}$-module V, the space $\mathbb{M}(V)$ is a $\mathfrak{g}\oplus\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})'$.*
(2) *For any $\varphi\in\mathcal{U}(\mathfrak{g})'$, there exists a $\mathfrak{g}$-module V, such that $\varphi\in\mathbb{M}(V)$. Moreover, if φ is $\mathfrak{n}_+\oplus\mathfrak{n}_+$-nilpotent, then V can be chosen from the category $\mathcal{O}$.*

Proof. To show that $\mathbb{M}(V)$ is invariant under the left action of $\mathfrak{g}$, we compute

$$(\pi_l(g)\phi_{v,v'})(x) = \phi_{v,v'}(xg) = \langle v', xg\, v\rangle = \phi_{gv,v'}(x),$$

for any $x\in\mathcal{U}(\mathfrak{g})$, $g\in\mathfrak{g}$, $v\in V$, $v'\in V'$. This shows that $y\phi_{v,v'}\in\mathbb{M}(V)$. The invariance under the right action follows from the computation

$$(\pi_r(g)\phi_{v,v'})(x) = -\phi_{v,v'}(\omega(g)x) = -\langle v', \omega(g)x\, v\rangle = \langle yv', x\, v\rangle = \phi_{v,yv'}(x).$$

For the second part, assume $\varphi\in\mathcal{U}(\mathfrak{g})'$. Denote by V the subspace of $\mathcal{U}(\mathfrak{g})'$, generated from φ by the left action of $\mathfrak{g}$. Let φ' be the restriction to V of the unit $1\in\mathcal{U}(\mathfrak{g})=\mathcal{U}(\mathfrak{g})''$. Equivalently, φ' is the linear functional on V, determined by $\langle\varphi',\psi\rangle=\psi(1)$, for any $\psi\in V\subset\mathcal{U}(\mathfrak{g})'$. We claim that $\varphi=\phi_{\varphi,\varphi'}\in\mathbb{M}(V)$. Indeed, for any $x\in\mathcal{U}(\mathfrak{g})$ we have

$$\phi_{\varphi,\varphi'}(x) = \langle\varphi', x\varphi\rangle = (x\varphi)(1) = \varphi(x).$$

Finally, if φ is left-$\mathfrak{n}_+$-nilpotent, then V is locally $\mathfrak{n}_+$-nilpotent. Since V is generated by a single element φ, it belongs to category $\mathcal{O}$. The right-$\mathfrak{n}_+$-nilpotency condition guarantees that φ' belongs to the *restricted* dual space V'. $\square$

The elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ may be regarded as the differential operators, acting on $\mathfrak{R}(G)$. This gives an interpretation of the regular functions on G (or even on G_0) as linear functionals on $\mathcal{U}(\mathfrak{g})$, and thus to identifications of the spaces $\mathfrak{R}(G)$ and $\mathfrak{R}(G_0)$ with certain subspaces of $\mathcal{U}(\mathfrak{g})'$. In the explicit realizations $\mathbf{F}$ and $\mathbb{F}$ this correspondence is constructed using the algebraic analogue of the "co-unit" element of the Hopf algebra $\mathfrak{R}(G)$ - the linear functional $\langle\cdot\rangle:\mathbb{F}\to\mathbb{C}$, defined by

$$\langle\beta^m\bar\beta^n\mathbf{1}_\lambda\rangle = \delta_{m,0}\delta_{n,0}. \tag{1.19}$$

Proposition 1.6. *The linear map $\vartheta:\mathbb{F}\to\mathcal{U}(\mathfrak{g})'$, defined by $v\mapsto\vartheta_v$,*

$$\vartheta_v(x) = \langle\pi_l(x)v\rangle, \qquad v\in\mathbb{F},\ x\in\mathcal{U}(\mathfrak{g}). \tag{1.20}$$

is an injective $\mathfrak{g}\oplus\mathfrak{g}$-homomorphism.

Proof. In terms of the polynomial realization, $\langle\cdot\rangle$ corresponds to evaluating a function $\psi(x,y,\zeta)\in\mathfrak{R}(G_0)$ at the element $\mathbf{w_0}$: $\langle\psi\rangle=\psi(0,0,1)$. This implies that for any $v\in\mathbb{F}$

$$\langle\mathbf{e}v\rangle = -\langle\bar{\mathbf{f}}v\rangle, \qquad \langle\mathbf{h}v\rangle = -\langle\bar{\mathbf{h}}v\rangle, \qquad \langle\mathbf{f}v\rangle = -\langle\bar{\mathbf{e}}v\rangle. \tag{1.21}$$

Therefore, for any $g\in\mathfrak{g}$ and $x\in\mathcal{U}(\mathfrak{g})$ we have

$$\vartheta_{gv}(x) = \langle x\, gv\rangle = \vartheta_v(xg) = (\pi_l(g)\vartheta_v)(x),$$

$$\vartheta_{\bar g v}(x) = \langle x\,\bar g v\rangle = \langle\bar g\, xv\rangle = -\langle\omega(g)xv\rangle = -\vartheta_v(\omega(g)x) = (\pi_r(g)\vartheta_v)(x).$$

We conclude that the map ϑ is a $\mathfrak{g}\oplus\mathfrak{g}$-homomorphism. To prove that it is injective, we need to show that for any nonzero $v\in\mathbb{F}$ there exists an element $x\in\mathfrak{g}\oplus\mathfrak{g}$ such that $\langle xv\rangle\neq 0$.

Since $\mathbb{F}$ is locally $\mathfrak{n}_+$-nilpotent, we can pick $k \geq 0$ such that $\mathbf{e}^k v \neq 0$, but $\mathbf{e}^{k+1} v = 0$. Replacing v by $\mathbf{e}^k v$, we see that it suffices consider the case of $v \neq 0$ such that $\mathbf{e} v = 0$. Similarly, we may assume that $\bar{\mathbf{e}} v = 0$.

A vector v satisfying $\mathbf{e} v = 0 = \bar{\mathbf{e}} v$ must have the form $v = \sum_{\lambda \in \mathbf{P}} c_\lambda \mathbf{1}_\lambda$ with only finitely many $c_\lambda \neq 0$. Using the formula for the Vandermonde determinant and the fact that

$$\langle \mathbf{h}^m v \rangle = \sum_{\lambda \in \mathbf{P}} c_\lambda \lambda^m, \qquad m \geq 0,$$

we conclude that $\langle \mathbf{h}^k v \rangle = 0$ for all $k \geq 0$ if and only if all c_λ vanish. Thus, $\theta_v = 0$ is equivalent to $v = 0$, which means that ϑ is an injection. $\square$

The following statement is an algebraic version of the classical Peter-Weyl theorem.

Theorem 1.7. *The space $\mathfrak{R}(G)$ of regular functions on G is spanned by the matrix elements of finite-dimensional irreducible $\mathfrak{g}$-modules,*

$$\mathfrak{R}(G) \cong \bigoplus_{\lambda \in \mathbf{P}^+} \mathbb{M}(V_\lambda).$$

The decomposition of $\mathfrak{R}(G)$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module is given by

$$\mathfrak{R}(G) \cong \bigoplus_{\lambda \in \mathbf{P}^+} V_\lambda \otimes V_\lambda^\star.$$

Remark 3. The subspace of $\mathcal{U}(\mathfrak{g})'$, corresponding to $\mathfrak{R}(G)$, is invariantly characterized as the restricted Hopf dual $\mathcal{U}(\mathfrak{g})'_{Hopf} \subset \mathcal{U}(\mathfrak{g})'$, defined by

$$\mathcal{U}(\mathfrak{g})'_{Hopf} = \{\phi \in \mathcal{U}(\mathfrak{g})' \big| \exists \text{ two-sided ideal } J \subset \mathcal{U}(\mathfrak{g}) \text{ such that } \phi(J) = 0 \text{ and } \operatorname{codim} J < \infty\}.$$

The extended space $\mathfrak{R}(G_0)$ corresponds to a larger subalgebra of $\mathcal{U}(\mathfrak{g})'$, spanned by the matrix elements of all modules in the category $\mathcal{O}$.

Recall that the category $\mathcal{O}$ has enough projectives; we denote by P_λ the indecomposable projective cover of the irreducible module V_λ. It is known that every indecomposable module in the category $\mathcal{O}$ with integral weights is isomorphic to a subfactor of the projective module, corresponding to some anti-dominant integral weight λ. In particular, this means that it suffices to consider the matrix elements of the big projective modules $\{P_\lambda\}_{\lambda<0}$.

The following result can be regarded as a non-semisimple generalization of the Peter-Weyl theorem.

Theorem 1.8. *The space $\mathfrak{R}(G_0)$ of regular functions on G_0 is spanned by the matrix elements of all big projective modules in the category $\mathcal{O}$,*

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda \in \mathbf{P}^+} \mathbb{M}(P_\lambda).$$

As a $\mathfrak{g} \oplus \mathfrak{g}$-module, $\mathfrak{R}(G_0)$ is given by

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda \in -\mathbf{P}^{++}} (P_\lambda \otimes P_\lambda^\star) \Big/ I_\lambda$$

where I_λ's are the $\mathfrak{g} \oplus \mathfrak{g}$-submodules of $P_\lambda \otimes P_\lambda^\star$, corresponding to identically vanishing matrix elements.

Proof. We use the realization of $\mathfrak{R}(G_0)$ in the Fock space $\mathbb{F}$. The inclusion (1.20) provides the identification of $\mathbb{F}$ with a subspace of $\mathcal{U}(\mathfrak{g})'$. Since $\mathbb{F}$ is locally $\mathfrak{n}_+ \oplus \mathfrak{n}_+$-nilpotent, Proposition 1.5 implies that for any $v \in \mathbb{F}$ there exists a $\mathfrak{g}$-module $W \in \mathcal{O}$ such that $\vartheta_v \in \mathbb{M}(W)$.

Let $W = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ be the decomposition of W into a direct sum of indecomposable submodules. Each indecomposable component W_i, $i = 1, \ldots, m$, is a subfactor of some big projective module P_{λ_i}. Then $\mathbb{M}(W_i) \subset \mathbb{M}(P_{\lambda_i})$, and therefore we have

$$\mathbb{M}(W) = \mathbb{M}(W_1) + \mathbb{M}(W_2) + \cdots + \mathbb{M}(W_m) \subset \bigoplus_{\lambda \in -\mathbf{P}^{++}} \mathbb{M}(P_\lambda),$$

which shows that $\vartheta(\mathbb{F}) \subset \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}(P_\lambda)$. To prove that in fact $\vartheta(\mathbb{F}) = \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}(P_\lambda)$, we compare the characters of the two spaces, and show that they have the same size.

For any $\lambda \in \mathbf{P}^+$ the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{M}(P_{-\lambda-2})$ is isomorphic to the quotient of the product $P_{-\lambda-2} \otimes P^\star_{-\lambda-2}$ by the kernel of the map

$$\Theta_\lambda : P_{-\lambda-2} \otimes P^\star_{-\lambda-2} \to \mathcal{U}(\mathfrak{g})', \qquad \Theta_\lambda(v \otimes v') = \phi_{v,v'}. \tag{1.22}$$

Obviously, $I_\lambda = \ker \Theta_\lambda$ is a $\mathfrak{g}\oplus\mathfrak{g}$-submodule of $P_{-\lambda-2} \otimes P^\star_{-\lambda-2}$; we describe it more explicitly. It is known that the module $P_{-\lambda-2}$ has a filtration $0 \subset P^{(0)} \subset P^{(1)} \subset P_{-\lambda-2}$ such that

$$P^{(0)} \cong V_{-\lambda-2}, \qquad P^{(1)}/P^{(0)} \cong V_\lambda, \qquad P_{-\lambda-2}/P^{(1)} \cong V_{-\lambda-2},$$

and the dual filtration of the module $P^\star_{-\lambda-2}$ is given by

$$0 \subset \operatorname{Ann}(P^{(1)}) \subset \operatorname{Ann}(P^{(0)}) \subset P^\star_{-\lambda-2}.$$

They determine a filtration of the tensor product

$$\begin{aligned} 0 \subset P^{(0)} \otimes \operatorname{Ann}(P^{(1)}) &\subset P^{(0)} \otimes \operatorname{Ann}(P^{(0)}) + P^{(1)} \otimes \operatorname{Ann}(P^{(1)}) \subset \\ &\subset P^{(0)} \otimes P^\star_{-\lambda-2} + P^{(1)} \otimes \operatorname{Ann}(P^{(0)}) + P_{-\lambda-2} \otimes \operatorname{Ann}(P^{(1)}) \subset \\ &\subset P^{(1)} \otimes P^\star_{-\lambda-2} + P_{-\lambda-2} \otimes \operatorname{Ann}(P^{(0)}) \subset P_{-\lambda-2} \otimes P^\star_{-\lambda-2}. \end{aligned}$$

If $v \in P^{(0)}$ and $v' \in \operatorname{Ann}(P^{(0)})$, then $\phi_{v,v'}$ is the zero functional, since for any $x \in \mathcal{U}(\mathfrak{g})$ we have $x\,v \in P^{(0)}$ and $\phi_{v,v'}(x) = \langle v', x\,v\rangle = 0$. Hence the submodule $P^{(0)} \otimes \operatorname{Ann}(P^{(0)})$ lies in the kernel of the map Θ_λ, and similarly does $P^{(1)} \otimes \operatorname{Ann}(P^{(1)})$. One can easily see that

$$\Theta_\lambda\left(P^{(1)} \otimes \operatorname{Ann}(P^{(0)})\right) = \mathbb{M}(V_\lambda),$$
$$\Theta_\lambda\left(P^{(0)} \otimes P^\star_{-\lambda-2}\right) = \Theta_\lambda\left(P_{-\lambda-2} \otimes \operatorname{Ann}(P^{(1)})\right) = \mathbb{M}(V_{-\lambda-2}).$$

It follows that the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{M}(P_{-\lambda-2})$ has a filtration

$$0 \subset \mathbb{M}^{(0)} \subset \mathbb{M}^{(1)} \subset \mathbb{M}^{(2)} = \mathbb{M}(P_{-\lambda-2})$$

such that

$$\begin{aligned} \mathbb{M}^{(2)}/\mathbb{M}^{(1)} &\cong V_{-\lambda-2} \otimes V^\star_{-\lambda-2}, \\ \mathbb{M}^{(1)}/\mathbb{M}^{(0)} &\cong (V_\lambda \otimes V^\star_{-\lambda-2}) \oplus (V_{-\lambda-2} \otimes V^\star_\lambda), \\ \mathbb{M}^{(0)} &\cong (V_\lambda \otimes V^\star_\lambda) \oplus (V_{-\lambda-2} \otimes V^\star_{-\lambda-2}). \end{aligned}$$

Thus, the block $\mathbb{F}(\lambda)$ of (1.15) is identified with the subspace, spanned by the matrix elements of the big projective module $P_{-\lambda-2}$. Taking direct sums over all $\lambda \in \mathbf{P}^+$, adding the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{M}(P_{-1}) \cong V_{-1} \otimes V^\star_{-1}$, and comparing with Theorem 1.3, we see that $\bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}(P_\lambda)$ and $\mathbb{F}$ have the same characters. The statement of the theorem follows. $\square$

1.5. **Cohomology of $\mathfrak{g}$ with coefficients in regular representations.** The algebra $\mathfrak{R}(G)$ contains the subalgebra $\mathfrak{R}(G)^G$ of the conjugation-invariant functions on G, which is linearly spanned by the characters of the irreducible finite-dimensional representations. The subalgebra $\mathfrak{R}(G)^G$ is thus isomorphic to the Grothendieck ring of the finite-dimensional representations of G.

There is an isomorphism $\mathfrak{R}(G)^G \cong \mathbb{C}[\mathbf{P}]^W$, obtained by restricting the group characters to $\mathfrak{h}$ and taking its Fourier expansion. Finally, the algebra $\mathfrak{R}(G)^G$ also admits a cohomological interpretation, which will be instrumental for further generalizations to the regular representations of the affine and Virasoro algebras. We briefly recall the definition of the cohomology of $\mathfrak{g}$.

Proposition 1.9. *Let $\mathbf{\Lambda} = \bigwedge \mathfrak{g}'$ be the exterior algebra of $\mathfrak{g}'$ with unit $\mathbf{1}$. Then*

(1) *The Clifford algebra, generated by $\{\iota(g),\, \varepsilon(g')\}_{g\in\mathfrak{g}, g'\in\mathfrak{g}'}$ with relations*

$$\{\iota(x), \iota(y)\} = \{\varepsilon(x'), \varepsilon(y')\} = 0, \qquad \{\iota(x), \varepsilon(y')\} = \langle y', x\rangle, \tag{1.23}$$

acts irreducibly on $\mathbf{\Lambda}$, so that for any $\omega \in \mathbf{\Lambda}$ we have

$$\iota(g)\mathbf{1} = 0, \qquad \varepsilon(g')\omega = g' \wedge \omega, \qquad g \in \mathfrak{g},\ g' \in \mathfrak{g}',\ \omega \in \mathbf{\Lambda}.$$

(2) *$\mathbf{\Lambda}$ is a commutative superalgebra,*

$$\omega_1 \wedge \omega_2 = (-1)^{|\omega_1|\cdot|\omega_2|}\, \omega_2 \wedge \omega_1, \qquad \omega_1, \omega_2 \in \mathbf{\Lambda},$$

where $|\cdot|$ is the natural grading on $\mathbf{\Lambda}$ satisfying $|\mathbf{1}| = 0$, $|\iota(g)| = -1$, $|\varepsilon(g')| = 1$.

(3) *The $\mathfrak{g}$-module structure on $\mathbf{\Lambda}$ is given by*

$$\pi_{\mathbf{\Lambda}}(x) = \sum_i \varepsilon(g_i')\iota([g_i, x]),$$

where $\{g_i\}$ is any basis of $\mathfrak{g}$, and $\{g_j'\}$ is the corresponding dual basis of $\mathfrak{g}'$.

Definition 1. The cohomology $H^\bullet(\mathfrak{g}; V)$ of $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module V is the cohomology of the graded complex $C^\bullet(\mathfrak{g}; V) = \mathbf{\Lambda}^\bullet \otimes V$, with the differential

$$\mathbf{d} = \sum_i \varepsilon(g_i')\pi_V(g_i) - \frac{1}{2}\sum_{i,j} \varepsilon(g_i')\varepsilon(g_j')\iota([g_i, g_j]), \tag{1.24}$$

where $\{g_i\}$ is any basis of $\mathfrak{g}$, and $\{g_i'\}$ is the dual basis of $\mathfrak{g}'$.

The following is one of the fundamental results in Lie algebra cohomology, (see e.g. [HS]).

Theorem 1.10. *For any finite-dimensional $\mathfrak{g}$-module V we have*

$$H^\bullet(\mathfrak{g}; V) \cong V^{\mathfrak{g}} \otimes H^\bullet_{DR}(G), \tag{1.25}$$

where $H^\bullet_{DR}(G)$ denotes the holomorphic de Rham cohomology $H^\bullet_{DR}(G)$ of the Lie group G.

If V is a commutative algebra with a compatible $\mathfrak{g}$-action, then its cohomology inherits the multiplication from V and $\mathbf{\Lambda}$, and $H^\bullet(\mathfrak{g}; V)$ becomes itself a commutative superalgebra. Moreover, the isomorphism (1.25) becomes an isomorphism of superalgebras, with respect to the cup product in $H^\bullet_{DR}(G)$.

The diagonal $\mathfrak{g}$-action in $\mathbf{F}$ corresponds to the coadjoint action of G in $\mathfrak{R}(G)$; thus, we get

Corollary 1.11. *There is an isomorphism of commutative superalgebras*

$$H^\bullet(\mathfrak{g}; \mathbf{F}) = \mathbb{C}[\mathbf{P}]^W \otimes H^\bullet_{DR}(G).$$

Our next goal is to study the cohomology of $\mathfrak{g}$ with coefficients in the extended regular representation $\mathbb{F} \cong \mathfrak{R}(G_0)$. For infinite-dimensional $\mathfrak{g}$-modules Theorem 1.10 does not hold, and the cohomology $H^\bullet(\mathfrak{g};\mathbb{F})$ does not reduce to $\mathbb{F}^{\mathfrak{g}} \otimes H^\bullet_{DR}(G)$. We have instead

Theorem 1.12. *There is an isomorphism of commutative superalgebras*

$$H^\bullet(\mathfrak{g};\mathbb{F}) \cong \mathbb{C}[\mathbf{P}]^W \otimes \bigwedge\nolimits^\bullet \mathbb{C}^2.$$

Proof. It is easy to show using the results of [W] that for $\lambda \geq -1$

$$H^n(\mathfrak{g}; V_\lambda \otimes V^\star_{-\lambda-2}) = H^n(\mathfrak{g}; V_{-\lambda-2} \otimes V^\star_\lambda) = \begin{cases} \mathbb{C}, & n = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

and that for $\lambda \geq 0$ we have $H^n(\mathfrak{g}; V_{-\lambda-2} \otimes V^\star_{-\lambda-2}) = 0$ for all n. The spectral sequence associated with the filtration of Theorem 1.3 can be used to show that

$$H^n(\mathfrak{g};\mathbb{F}(-1)) = \begin{cases} \mathbb{C}, & n = 1,2 \\ 0, & \text{otherwise} \end{cases}, \qquad H^n(\mathfrak{g};\mathbb{F}(\lambda)) = \begin{cases} \mathbb{C}, & n = 0,2 \\ \mathbb{C}^2, & n = 1 \\ 0, & \text{otherwise} \end{cases}, \quad \lambda \geq 0. \tag{1.26}$$

Also, this spectral sequence shows that we have a natural isomorphism $H^0(\mathfrak{g};\mathbb{F}) \cong H^0(\mathfrak{g};\mathbf{F})$.

To explicitly get the generators of the commutative superalgebra $H^\bullet(\mathfrak{g};\mathbb{F})$, we pick nonzero elements

$$\chi \in H^0(\mathfrak{g};\mathbb{F}(1)), \qquad \xi_{-1} \in H^1(\mathfrak{g};\mathbb{F}(-1)), \qquad \eta_0 \in H^1(\mathfrak{g};\mathbb{F}(0)),$$

such that η_0 is not proportional to $\chi\,\xi_{-1}$. It is known that $H^0(\mathfrak{g};\mathbb{F}) \cong \mathbb{C}[\mathbf{P}]^W$ is isomorphic to the polynomial algebra $\mathbb{C}[\chi]$. It is also clear that $H^\bullet(\mathfrak{g};\mathbb{F})$ is a free $\mathbb{C}[\chi]$-module. For each $\lambda \geq 0$, the set

$$B_{\leq\lambda} = \{\xi_{-1}, \chi\xi_{-1}, \ldots, \chi^{\lambda+1}\,\xi_{-1}\} \bigcup \{\eta_0, \chi\,\eta_0, \ldots, \chi^\lambda\,\eta_0\}$$

consists of $2\lambda+3$ linearly independent elements, and in view of (1.26) is a basis of $H^1(\mathfrak{g};\mathbb{F}_{\leq\lambda})$. Finally, one can check that $\eta_0\,\xi_{-1} \neq 0$, and thus the elements $\{\eta_0\,\xi_{-1}, \chi\,\eta_0\,\xi_{-1}, \ldots, \chi^{\lambda+1}\,\eta_0\,\xi_{-1}\}$ give a basis of $H^2(\mathfrak{g};\mathbb{F}_{\leq\lambda})$ for each $\lambda \geq -1$.

It follows that $H^\bullet(\mathfrak{g};\mathbb{F}) \cong \mathbb{C}[\chi] \otimes \bigwedge^\bullet[\xi_{-1}, \eta_0]$, and the theorem is proven. □

Remark 4. One of the ingredients in the exterior algebra part of the cohomology $H^\bullet(\mathfrak{g};\mathbb{F})$ is the exterior algebra $\bigwedge^\bullet \mathfrak{h}$, corresponding to $\bigwedge^\bullet[\eta_0]$ above. It would be interesting to obtain an invariant characterization of the remaining part of $H^\bullet(\mathfrak{g};\mathbb{F})$ for arbitrary $\mathfrak{g}$.

Remark 5. In each of the two-dimensional spaces $H^1(\mathfrak{g};\mathbb{F}(\lambda))$ there is a unique up to proportionality cohomology class ξ_λ divisible by ξ_{-1}; the elements $\frac{\xi_\lambda}{\xi_{-1}}$ constitute a basis of $H^0(\mathfrak{g};\mathbb{F}) \cong \mathbb{C}[\mathbf{P}]^W$, associated with the characters of big projective modules (cf. [La]).

2. Modified regular representations of the affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$.

2.1. **Regular representations of $\widehat{\mathfrak{sl}}(2,\mathbb{C})$.** Let $\hat{G}$ be the central extension of the loop group LG, associated with $G = SL(2,\mathbb{C})$ (see [PS]), and let $\hat{\mathfrak{g}}$ be the corresponding Lie algebra. As we discussed in the introduction, there is no maximal cell in the affine Bruhat decomposition, and thus we will use the loop version (0.7) of the finite-dimensional one. An additional advantage is that we get an explicit realization of the left and right regular $\hat{\mathfrak{g}}$-actions, analogous to the finite-dimensional case.

The standard basis of $\hat{\mathfrak{g}}$ consists of the elements $\{\mathbf{e}_n, \mathbf{h}_n, \mathbf{f}_n\}_{n\in\mathbb{Z}}$ and the central element $\mathbf{k}$, subject to the commutation relations

$$[\mathbf{h}_m, \mathbf{e}_n] = 2\mathbf{e}_{m+n}, \qquad [\mathbf{h}_m, \mathbf{f}_n] = -2\mathbf{f}_{m+n}, \qquad [\mathbf{h}_m, \mathbf{h}_n] = 2m\delta_{m+n,0}\mathbf{k},$$
$$[\mathbf{e}_m, \mathbf{f}_n] = \mathbf{h}_{m+n} + m\,\delta_{m+n,0}\mathbf{k}, \qquad [\mathbf{e}_m, \mathbf{e}_n] = [\mathbf{f}_m, \mathbf{f}_n] = 0.$$

The Lie algebra $\hat{\mathfrak{g}}$ has a $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{g}}[n]$, determined by

$$\deg \mathbf{f}_n = \deg \mathbf{h}_n = \deg \mathbf{e}_n = -n, \qquad \deg \mathbf{k} = 0,$$

We introduce subalgebras $\hat{\mathfrak{g}}_\pm = \bigoplus_{\pm n>0} \mathfrak{g}[n]$; the finite-dimensional Lie algebra $\mathfrak{g}$ is naturally identified with a subalgebra in $\hat{\mathfrak{g}}[0]$.

The element $\mathbf{w_0}$ of the classical Weyl group defines an involution $\hat{\omega}$ of $\hat{\mathfrak{g}}$, such that

$$\hat{\omega}(\mathbf{e}_n) = -\mathbf{f}_n, \quad \hat{\omega}(\mathbf{h}_n) = -\mathbf{h}_n, \quad \hat{\omega}(\mathbf{f}_n) = -\mathbf{e}_n, \quad \hat{\omega}(\mathbf{k}) = -\mathbf{k}. \tag{2.1}$$

We use the loop version of the finite-dimensional Bruhat decomposition (0.4), and factorize the central extension $\widehat{LT}$ into the product of loops that extend holomorphically inside and outside of the unit circle. The analogue of (1.2) is the formal decomposition

$$g = \exp\left(\sum_{n\in\mathbb{Z}} x_n \mathbf{e}_n\right) \ \mathbf{w_0}\,\tau^{\mathbf{k}} \ \exp\left(\sum_{m<0} \zeta_m \mathbf{h}_m\right) \zeta^{\mathbf{h}_0} \exp\left(\sum_{m>0} \zeta_m \mathbf{h}_m\right) \exp\left(\sum_{n\in\mathbb{Z}} y_n \mathbf{e}_n\right).$$

The polynomial algebra $\mathfrak{R}_0(\hat{G}_0) = \mathbb{C}[\{x_n\}, \{y_n\}, \{\zeta_{n\neq 0}\}, \zeta^{\pm 1}]$ can be thought of as the algebra of regular functions on the big cell of the loop group, and for $\mathfrak{R}(\hat{G}_0)$ we get

$$\mathfrak{R}(\hat{G}_0) = \mathfrak{R}_0(\hat{G}_0) \otimes \mathbb{C}[\tau^{\pm 1}] = \bigoplus_{\varkappa\in\mathbb{Z}} \mathfrak{R}_\varkappa(\hat{G}_0), \qquad \mathfrak{R}_\varkappa(\hat{G}_0) = \mathfrak{R}_0(\hat{G}_0) \otimes \mathbb{C}\tau^{\varkappa}$$

Note that for each $\varkappa$ the subspace $\mathfrak{R}_\varkappa(\hat{G}_0)$ is a $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$-submodule of $\mathfrak{R}(\hat{G}_0)$, but it is not a subalgebra of $\mathfrak{R}(\hat{G}_0)$ when $\varkappa \neq 0$! It is easy to see that the infinitesimal regular $\hat{\mathfrak{g}}$-actions of the central element $\mathbf{k}$ on $\mathfrak{R}_\varkappa(\hat{G}_0)$ are given by

$$\pi_l(\mathbf{k}) = -\varkappa \cdot \mathrm{Id}, \qquad \pi_r(\mathbf{k}) = \varkappa \cdot \mathrm{Id}\,. \tag{2.2}$$

As vector spaces, all $\mathfrak{R}_\varkappa(\hat{G}_0)$ are identified with the same polynomial space, and one can compute the infinitesimal regular actions of $\hat{\mathfrak{g}}$ by treating $\varkappa$ as a complex parameter. In particular, the regular actions of $\hat{\mathfrak{g}}$ make sense for arbitrary $\varkappa \in \mathbb{C}$. Computations yield the following description, analogous to Proposition 1.1.

Theorem 2.1. *The regular actions of $\hat{\mathfrak{g}}$ on $\mathfrak{R}_\varkappa(\hat{G}_0)$ are given by* (2.2) *and*

$$\begin{aligned}
&\pi_l(\mathbf{e}_n) = -\partial_{x_n}, \\
&\pi_l(\mathbf{h}_n) = -2\sum_{i\in\mathbb{Z}} x_i \partial_{i_{n+n}} + \begin{cases} \partial_{\zeta_n} + 2n\varkappa\,\zeta_{-n}, & n>0 \\ \zeta\,\partial_\zeta, & n=0\ , \\ \partial_{\zeta_n}, & n<0 \end{cases} \\
&\pi_l(\mathbf{f}_n) = \sum_{i,i'\in\mathbb{Z}} x_i x_{i'} \partial_{x_{i+i'+n}} - \sum_{j<0} x_{j-n}\partial_{\zeta_j} - x_{-n}\zeta\,\partial_\zeta - \sum_{j>0} x_{j-n}\left(\partial_{\zeta_j} + 2j\varkappa\,\zeta_{-j}\right) - \\
&\qquad - \varkappa\, n x_{-n} + \zeta^{-2} \sum_{j,j'>0} \mathrm{s}_{j'}(-2\zeta_1, -2\zeta_2, \dots)\, \mathrm{s}_j(-2\zeta_{-1}, -2\zeta_{-2}, \dots) \partial_{y_{n-j+j'}},
\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&\pi_r(\mathbf{e}_n) = \partial_{y_n},\\
&\pi_r(\mathbf{h}_n) = -2\sum_{i\in\mathbb{Z}} y_i\partial_{y_{i+n}} + \begin{cases}\partial_{\zeta_n} & n>0,\\ \zeta\,\partial_\zeta, & n=0,\ ,\\ \partial_{\zeta_n} - 2n\varkappa\,\zeta_{-n}, & n<0.\end{cases}\\
&\pi_r(\mathbf{f}_n) = -\sum_{i,i'\in\mathbb{Z}} y_i y_{i'}\partial_{y_{i+i'+n}} + \sum_{j>0} y_{j-n}\partial_{\zeta_j} + y_{-n}\zeta\,\partial_\zeta + \sum_{j<0} y_{j-n}\left(\partial_{\zeta_j} - 2j\varkappa\,\zeta_{-j}\right) -\\
&\qquad - \varkappa\, n y_{-n} - \zeta^{-2}\sum_{j,j'>0} \mathrm{s}_{j'}(-2\zeta_{-1},-2\zeta_{-2},\dots)\,\mathrm{s}_j(-2\zeta_1,-2\zeta_2,\dots)\partial_{x_{n+j-j'}},
\end{aligned}
\tag{2.4}
$$

where the Schur polynomials $\mathrm{s}_k(\alpha_1,\alpha_2,\dots)$ *are defined by*

$$
\mathrm{s}_m(\alpha_1,\alpha_2,\dots) = \sum_{\substack{l_1,l_2,\dots\geq 0\\ l_1+2l_2+\dots=m}} \frac{\alpha_1^{l_1}\alpha_2^{l_2}\dots}{l_1!l_2!\dots}.
$$

Proof. The presence of the central extension requires the use of some elementary cases of the Campbell-Hausdorff formula in our computations; we use the identity

$$
\exp(B)\exp(tA) \overset{\mathrm{mod}\ t^2}{\equiv} \exp\left(t\sum_{j=0}^{\infty}\frac{1}{j!}\underbrace{[B,\dots,[B,[B,A]]\dots]}_{j\text{ commutators}}\right)\exp(B).
$$

For example, to derive the last of (2.4), we use the formulas:

$$
\begin{aligned}
\exp\left(\sum_{i\in\mathbb{Z}} y_i\mathbf{e}_i\right)\exp(t\mathbf{f}_n) &\overset{\mathrm{mod}\ t^2}{\equiv} \exp(t\mathbf{f}_n)\exp\left(-tny_{-n}\mathbf{k} + t\sum_{i\in\mathbb{Z}} y_i\mathbf{h}_{i+n}\right)\times\\
&\qquad\times\exp\left(-t\sum_{i,i'\in\mathbb{Z}} y_i y_{i'}\mathbf{e}_{i+i'+n}\right)\exp\left(\sum_{i\in\mathbb{Z}} y_i\mathbf{e}_i\right),\\
\exp\left(\sum_{m>0}\zeta_m\mathbf{h}_m\right)\exp(t\mathbf{f}_n) &\overset{\mathrm{mod}\ t^2}{\equiv} \exp\left(t\sum_{j>0}\mathrm{s}_j(-2\zeta_1,-2\zeta_2,\dots)\mathbf{f}_{n+j}\right)\exp\left(\sum_{m>0}\zeta_m\mathbf{h}_m\right),\\
\zeta^{\mathbf{h}_0}\exp(t\mathbf{f}_n) &\overset{\mathrm{mod}\ t^2}{\equiv} \exp\left(t\zeta^{-2}\,\mathbf{f}_n\right)\zeta^{\mathbf{h}_0},\\
\exp\left(\sum_{m<0}\zeta_m\mathbf{h}_m\right)\exp(t\mathbf{f}_n) &\overset{\mathrm{mod}\ t^2}{\equiv} \exp\left(t\sum_{j'>0}\mathrm{s}_{j'}(-2\zeta_{-1},-2\zeta_{-2},\dots)\mathbf{f}_{n-j'}\right)\exp\left(\sum_{m<0}\zeta_m\mathbf{h}_m\right),\\
\mathbf{w_0}\exp(t\mathbf{f}_n) &\overset{\mathrm{mod}\ t^2}{\equiv} \exp(-t\,\mathbf{e}_n)\,\mathbf{w_0}.
\end{aligned}
$$

Combining these equations, we get the desired formulas. We leave the technical calculations to the reader. □

2.2. **Vertex operator algebras: review and useful examples.** We aim to endow $\mathfrak{R}_\varkappa(\hat{G}_0)$ (or its modification) with a structure similar to that of an associative commutative algebra. The relevant formalism is provided by the vertex algebra theory.

We recall the definitions of vertex and vertex operator algebras in the most convenient to us form. For more details and equivalent alternative definitions, we refer the reader to the books on the subject [FLM, FrB].

Let $\mathcal{V}$ be a vector space, equipped with a linear correspondence

$$v \mapsto \mathcal{Y}(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}, \qquad v_{(n)} \in \mathrm{End}(\mathcal{V}). \tag{2.5}$$

We refer to such formal $\mathrm{End}(\mathcal{V})$-valued generating functions as 'quantum fields'.

We say that $\mathcal{V}$ satisfies the locality property, if for any $a, b \in \mathcal{V}$

$$(z-w)^N [\mathcal{Y}(a,z), \mathcal{Y}(b,w)] = 0 \quad \text{for } N \gg 0 \tag{2.6}$$

in the ring of $\mathrm{End}(\mathcal{V})$-valued formal Laurent series in two variables z, w.

A vector $\mathbf{1} \in \mathcal{V}$ is called the vacuum vector, if it satisfies

$$\mathcal{Y}(\mathbf{1}, z) = \mathrm{Id}_{\mathcal{V}}, \qquad \mathcal{Y}(v,z)\mathbf{1}\Big|_{z=0} = v. \tag{2.7}$$

An element $\mathcal{D} \in \mathrm{End}(\mathcal{V})$, is called the infinitesimal translation operator, if it satisfies

$$\mathcal{D}\,\mathbf{1} = 0, \qquad [\mathcal{D}, \mathcal{Y}(v,z)] = \frac{d}{dz}\mathcal{Y}(v,z), \qquad \text{for all } \ v \in \mathcal{V}. \tag{2.8}$$

Definition 2. The space $\mathcal{V}$ is called a vertex algebra, if it is equipped with a linear map (2.5), vacuum vector $\mathbf{1}$, and infinitesimal translation operator $\mathcal{D}$, satisfying the axioms (2.6), (2.7), (2.8) above.

Vertex superalgebras are defined as usual by inserting $\pm$ signs according to parity. A vertex superalgebra $\mathcal{V}$ is called bi-graded, if it has $\mathbb{Z}$-gradings, $|\cdot|$ and deg,

$$\mathcal{V} = \bigoplus_{m,n\in\mathbb{Z}} \mathcal{V}^m[n], \qquad \mathcal{V}^m[n] = \left\{ v \in \mathcal{V} \,\middle|\, |v| = m \text{ and } \deg v = n \right\},$$

such that the parity in superalgebra is determined by $|\cdot|$, and for any homogeneous v

$$v \mapsto \mathcal{Y}(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}, \qquad \text{with } |v_{(n)}| = |v| \text{ and } \deg v_{(n)} = \deg v - n - 1.$$

In particular, for the vacuum we must have $|\mathbf{1}| = \deg \mathbf{1} = 0$. Also, we write $|\mathcal{Y}(v,z)| = |v|$ and $\deg \mathcal{Y}(v,z) = \deg v$ for the quantum field $\mathcal{Y}(v,z)$, if the above conditions are satisfied.

A vertex algebra $\mathcal{V}$ is called a vertex operator algebra (VOA) of rank $c \in \mathbb{C}$, if there exists an element $\boldsymbol{\omega} \in \mathcal{V}$, usually called the Virasoro element, such that the operators $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ defined by

$$\mathcal{Y}(\boldsymbol{\omega}, z) = \sum_{n\in\mathbb{Z}} \mathcal{L}_n z^{-n-2},$$

satisfy $\mathcal{L}_{-1} = \mathcal{D}$, and the Virasoro commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}\,c.$$

We define the the normal ordered product of two quantum fields $X(z)$ and $Y(z)$ by

$$:X(z)Y(w):= X_-(z)Y(w) + Y(w)X_+(z),$$

where $X_\pm(z)$ are the regular and principal parts of $X(z) = \sum_{n\in\mathbb{Z}} X_{(n)} z^{-n-1}$,

$$X_+(z) = \sum_{n\geq 0} X_{(n)} z^{-n-1}, \qquad X_-(z) = \sum_{n<0} X_{(n)} z^{-n-1}.$$

For products of three or more quantum fields, the normal ordered product is defined inductively, starting from the left. In general, the normal ordered product is neither commutative nor associative.

The following 'reconstruction theorem' is an effective tool for constructing vertex algebras.

Proposition 2.2. *Let $\mathcal{V}$ be a vector space with a distinguished vector $\mathbf{1}$ and a family of pairwise local* $\mathrm{End}(\mathcal{V})$*-valued quantum fields $\{X^\alpha(z) = \sum_{n\in\mathbb{Z}} X^\alpha_{(n)} z^{-n-1}\}_{\alpha\in\mathfrak{I}}$. Suppose $\mathcal{V}$ is generated from $\mathbf{1}$ by the action of the Laurent coefficients of quantum fields $X^\alpha(w)$, and that the vectors $\{X^\alpha(z)\mathbf{1}\big|_{z=0}\}_{\alpha\in\mathfrak{I}}$ are linearly independent in $\mathcal{V}$. Then the operators*

$$\mathcal{Y}\left(X^{\alpha_1}_{(-n_1-1)}\dots X^{\alpha_k}_{(-n_k-1)}\mathbf{1}, z\right) = :X^{\alpha_1}(z)^{(n_1)}\dots X^{\alpha_k}(z)^{(n_k)}:,$$

where $X(z)^{(n)} = \frac{1}{n!}\frac{d^n}{dz^n}X(z)$, satisfy (2.6) *and* (2.7).

If a linear operator $\mathcal{D}\in$ $\mathrm{End}(\mathcal{V})$ *satisfies $\mathcal{D}\mathbf{1} = 0$ and $[\mathcal{D}, X^\alpha(z)] = \frac{d}{dz}X^\alpha(z)$ for every $\alpha\in\mathfrak{I}$, then $[\mathcal{D}, \mathcal{Y}(v,z)] = \frac{d}{dz}\mathcal{Y}(v,z)$ for any $v\in\mathcal{V}$.*

We say that a vertex algebra $\mathcal{V}$ has a PBW basis, associated with quantum fields $\{X^\alpha(z)\}_{\alpha\in\mathfrak{I}}$, if the index set $\mathfrak{I}$ is ordered, and we have a linear basis of $\mathcal{V}$, formed by the vectors

$$\left\{X^{\alpha_1}_{(-n_1-1)}\dots X^{\alpha_k}_{(-n_k-1)}\mathbf{1}\;\middle|\; n_1\geq n_2\geq\dots\geq n_k\geq 0, \text{ and if } n_i = n_{i+1}, \text{ then } \alpha_i\preceq\alpha_{i+1}\right\}.$$

For two mutually local quantum fields $X(z), Y(w)$ we introduce the operator product expansion (OPE) formalism, and write

$$X(z)Y(w)\sim\sum_j\frac{C_j(w)}{(z-w)^j},$$

if for a finite collection of quantum fields $\{C_j(w)\}_{j=1,2,\dots}$ we have the equality

$$X(z)Y(w) = \sum_j\frac{C_j(w)}{(z-w)^j} + :X(z)Y(w):$$

where $\frac{1}{(z-w)^j}$ should be expanded into the Laurent series in non-negative powers of $\frac{w}{z}$. The importance of OPE lies in the fact that all commutators $[X_m, Y_n]$ of Laurent coefficients of quantum fields $X(z), Y(w)$ are completely encoded by the collection $\{C_j(w)\}$.

The remainder of this subsection presents some examples of vertex algebras, which will be used in this paper. All of these algebras are bi-graded and have a PBW basis associated with given quantum fields, for which we specify the OPEs.

Example 1. We denote by $\hat{F}(\beta,\gamma)$ the vertex algebra generated by quantum fields

$$\beta(z) = \sum_{n\in\mathbb{Z}}\beta_n z^{-n}, \qquad |\beta(z)| = 0, \qquad \deg\beta(z) = 0,$$
$$\gamma(z) = \sum_{n\in\mathbb{Z}}\gamma_n z^{-n-1}, \qquad |\gamma(z)| = 1, \qquad \deg\gamma(z) = 0,$$

with the operator product expansions

$$\beta(z)\gamma(w)\sim\frac{1}{z-w}, \qquad \beta(z)\beta(w)\sim\gamma(z)\gamma(w)\sim 0. \tag{2.9}$$

The commutation relations for the underlying Heisenberg algebra are

$$[\beta_m, \gamma_n] = \delta_{m+n,0}, \qquad [\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0. \tag{2.10}$$

Example 2. We denote by $\hat{\Lambda}(\psi, \psi^*)$ the vertex superalgebra generated by quantum fields

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}, \qquad |\psi(z)| = -1, \qquad \deg \psi(z) = 1,$$
$$\psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n}, \qquad |\psi^*(z)| = 1, \qquad \deg \psi^*(z) = 0,$$

with the operator product expansions

$$\psi(z)\psi(w) \sim \psi^*(z)\psi^*(w) \sim 0, \qquad \psi(z)\psi^*(w) \sim \frac{1}{z-w}.$$

The (anti)-commutation relations for the underlying Clifford algebra are

$$\{\psi_m, \psi_n\} = \{\psi_m^*, \psi_n^*\} = 0, \qquad \{\psi_m, \psi_n^*\} = \delta_{m+n,0}. \tag{2.11}$$

Example 3. We denote by $\hat{\mathfrak{g}}_k$ the vertex algebra generated by quantum fields

$$X_n = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}, \qquad |X(z)| = 0, \qquad \deg X(z) = 1, \qquad X \in \mathfrak{g},$$

with the operator product expansions

$$X(z)Y(w) \sim \frac{[X,Y](w)}{z-w} + k\,\frac{\langle X, Y\rangle}{(z-w)^2}, \qquad k \in \mathbb{C}$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$. The number k is called the level of $\hat{\mathfrak{g}}_k$.

We note that a module for the vertex algebra $\hat{\mathfrak{g}}_k$ is a $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module $\hat{V} = \bigoplus_{n\geq n_0} \hat{V}[n]$, such that $\pi_{\hat{V}}(\mathbf{k}) = k \cdot \mathrm{Id}_{\hat{V}}$ and $\hat{\mathfrak{g}}[m]\hat{V}[n] \subset \hat{V}[m+n]$ for any $m, n \in \mathbb{Z}$.

Example 4. We denote by Vir_c the vertex algebra generated by the quantum field

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad |L(z)| = 0, \qquad \deg L(z) = 2,$$

with the operator product expansion

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{z-w}, \qquad c \in \mathbb{C}.$$

The number c is called the central charge of Vir_c.

A module for the vertex algebra Vir_c is a $\mathbb{Z}$-graded Vir-module $\tilde{V} = \bigoplus_{n\geq n_0} \tilde{V}[n]$, such that $\pi_{\tilde{V}}(\mathbf{c}) = c \cdot \mathrm{Id}_{\tilde{V}}$ and $L_{-m}\tilde{V}[n] \subset \tilde{V}[m+n]$ for any $m, n \in \mathbb{Z}$.

Example 5. We denote by $\hat{F}_\varkappa(a)$ the vertex algebra generated by the quantum field

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad |a(z)| = 0, \qquad \deg a(z) = 1,$$

with the operator product expansion

$$a(z)a(w) \sim \frac{2\varkappa}{(z-w)^2}, \qquad \varkappa \in \mathbb{C}. \tag{2.12}$$

The commutation relations for the underlying Heisenberg algebra $\mathcal{H}(a)$ are

$$[a_m, a_n] = 2\varkappa\, m\, \delta_{m+n,0}. \tag{2.13}$$

Note that the operator a_0 is central and kills the vacuum.

Below we give the construction of a vertex algebra, which will be crucial for our future considerations. Let $\hat{F}_{-\varkappa}(\bar{a})$ be defined similarly to $\hat{F}_\varkappa(a)$, so that

$$[\bar{a}_m, \bar{a}_n] = -2\varkappa\, m\, \delta_{m+n,0}, \qquad \bar{a}(z)\bar{a}(w) \sim -\frac{2\varkappa}{(z-w)^2}. \tag{2.14}$$

Theorem 2.3. *Let $\varkappa \neq 0$. The space $\tilde{\mathbb{F}}_\varkappa = \hat{F}_\varkappa(a) \otimes \hat{F}_{-\varkappa}(\bar{a}) \otimes \mathbb{C}[\mathbf{P}]$ has a vertex algebra structure, extending those of $\hat{F}_\varkappa(a)$ and $\hat{F}_{-\varkappa}(\bar{a})$, and such that $a_0\mathbf{1}_\lambda = \bar{a}_0\mathbf{1}_\lambda = \lambda\mathbf{1}_\lambda$.*

Proof. Introduce the quantum fields $\{\mathbb{Y}(\mu, w)\}_{\mu\in\mathbf{P}}$ by

$$\begin{aligned}\mathbb{Y}(\mu, z) = \exp\left(\frac{\mu}{2\varkappa}\sum_{n<0}\frac{a_n}{-n}z^{-n}\right)\exp\left(\frac{\mu}{2\varkappa}\sum_{n>0}\frac{a_n}{-n}z^{-n}\right)\times \\ \times\exp\left(-\frac{\mu}{2\varkappa}\sum_{n<0}\frac{\bar{a}_n}{-n}z^{-n}\right)\exp\left(-\frac{\mu}{2\varkappa}\sum_{n>0}\frac{\bar{a}_n}{-n}z^{-n}\right)\mathbf{1}_\mu.\end{aligned} \tag{2.15}$$

Straightforward computations lead to the operator product expansions

$$a(z)\bar{a}(w) \sim \bar{a}(z)a(w) \sim \mathbb{Y}(\mu, z)\mathbb{Y}(\nu, w) \sim 0,$$

$$a(z)\mathbb{Y}(\mu, w) \sim \bar{a}(z)\mathbb{Y}(\mu, w) \sim \frac{\mu\,\mathbb{Y}(\mu, w)}{z-w},$$

and establish mutual pairwise locality for the quantum fields $a(z), \bar{a}(z), \mathcal{Y}(\mu, z)$.

The vacuum is, of course, the vector $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}_0 \in \mathbb{F}_\varkappa$. We set $\mathcal{Y}(\mathbf{1}_\lambda, z) = \mathbb{Y}(\lambda, z)$ for any $\lambda \in \mathbf{P}$. The spanning and linear independence conditions of Proposition 2.2 are immediate. Finally, we set $\mathcal{D}\mathbf{1}_\lambda = \frac{\lambda}{2\varkappa}(a_{-1} - \bar{a}_{-1})\mathbf{1}_\lambda$. The conditions on $\mathcal{D}$ amount to

$$\mathbb{Y}'(\lambda, z) = \frac{\lambda}{2\varkappa}\Big(:a(z)\mathbb{Y}(\lambda, z): - :\bar{a}(z)\mathbb{Y}(\lambda, z):\Big), \tag{2.16}$$

which is checked directly. Applying Proposition 2.2, we get the desired statement. $\square$

Theorem 2.3 should be compared with the construction of lattice vertex algebras. It is known that the space $\hat{F}_\varkappa(a) \otimes \mathbb{C}[\mathbf{P}]$ carries a vertex algebra structure only for special values of $\varkappa$, satisfying certain integrality conditions.

2.3. **Bosonic realizations.** We now proceed to study the generalizations of the algebra $\mathfrak{R}(G_0)$. As in the finite-dimensional case, we study modules for the Lie algebra $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$, which is equivalent to having two commuting actions of $\hat{\mathfrak{g}}$ on the same space.

As in the classical case, the regular $\hat{\mathfrak{g}}$-actions on $\mathfrak{R}_\varkappa(\hat{G}_0)$, described in Theorem 2.1, can be reformulated in terms of representations of Heisenberg algebras. We note that the operators

$$\begin{matrix} \beta_n = -y_{-n} \\ \gamma_n = \partial_{y_n} \end{matrix}, \qquad a_n = \begin{cases} \partial_{\zeta_n}, & n > 0 \\ \zeta\partial_\zeta, & n = 0 \\ \partial_{\zeta_n} - 2n\varkappa\,\zeta_{-n}, & n < 0 \end{cases}$$

satisfy the commutation relations (2.10),(2.13), and similarly for

$$\begin{matrix} \bar\beta_n = x_{-n}, \\ \bar\gamma_n = -\partial_{x_n} \end{matrix}, \qquad \bar a_n = \begin{cases} \partial_{\zeta_n} + 2n\varkappa\,\zeta_{-n} & n > 0 \\ \zeta\partial_\zeta, & n = 0\,. \\ \partial_{\zeta_n} & n < 0 \end{cases}$$

Note also that $\mathbb{C}[\zeta^{\pm 1}] \cong \mathbb{C}[\mathbf{P}]$, and $a_0 = \bar a_0 = a$ act on $\mathbb{C}[\mathbf{P}]$ by derivations $a\mathbf{1}_\lambda = \lambda\mathbf{1}_\lambda$.

The formulas of Theorem 2.1 are particularly simple, when written for the generating series $\mathbf{e}(z), \mathbf{h}(z), \mathbf{f}(z)$. For example, (2.4) becomes

$$\begin{aligned} &\pi_r(\mathbf{e}(z)) = \gamma(z), \\ &\pi_r(\mathbf{h}(z)) = 2\beta(z)\gamma(z) + a(z), \\ &\pi_r(\mathbf{f}(z)) = -\beta(z)^2\gamma(z) - \beta(z)a(z) - \varkappa\beta'(z) + \exp\left(\frac{1}{\varkappa}\sum_{n\neq 0}\frac{a_n - \bar a_n}{n} z^{-n}\right)\bar\gamma(z)\mathbf{1}_{-2}. \end{aligned} \tag{2.17}$$

Note that in this polynomial realization the constants are annihilated by $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ and $\{\mathbf{h}_n\}_{n\geq 0}$. The vertex algebra formalism requires a different choice of vacuum, and the introduction of normal ordering to make products of quantum fields well-defined. This procedure is well-known in the theory of Wakimoto modules (see [FrB] and references therein), for which the $\hat{\mathfrak{g}}$-action is constructed by modifying the formulas originating from the semi-infinite flag variety. In particular, one expects the shifts of the levels of the representations by the dual Coxeter number $h^\vee = 2$.

The modifications of the formulas (2.17) leads to the following result.

Theorem 2.4. *Let $\varkappa \neq 0$, and let $k = \varkappa - h^\vee$ and $\bar k = -\varkappa - h^\vee$, and let*

$$\hat{\mathbb{F}}_\varkappa = \hat F(\beta,\gamma) \otimes \hat F(\bar\beta,\bar\gamma) \otimes \tilde{\mathbb{F}}_\varkappa.$$

(1) *The space $\hat{\mathbb{F}}_\varkappa$ has a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar k}$-module structure, defined by*

$$\begin{aligned} &\mathbf{e}(z) = \gamma(z), \\ &\mathbf{h}(z) = 2 :\beta(z)\gamma(z): + a(z), \\ &\mathbf{f}(z) = - :\beta(z)^2\gamma(z): - \beta(z)a(z) - k\beta'(z) + \mathbb{Y}(-2,z)\bar\gamma(z), \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\bar{\mathbf{e}}(z) = \bar\gamma(z), \\ &\bar{\mathbf{h}}(z) = 2 :\bar\beta(z)\bar\gamma(z): + \bar a(z), \\ &\bar{\mathbf{f}}(z) = - :\bar\beta(z)^2\bar\gamma(z): - \bar\beta(z)\bar a(z) - \bar k\bar\beta'(z) + \mathbb{Y}(-2,z)\gamma(z). \end{aligned} \tag{2.19}$$

(2) *The space* $\hat{\mathbb{F}}_\varkappa$ *has a compatible VOA structure with* $\operatorname{rank}\hat{\mathbb{F}}_\varkappa = 6$. *(Compatible means that the operators* $\mathcal{Y}(v,z)$ *are* $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$*-intertwining operators in the VOA sense).*

Similar formulas for the two commuting actions of $\hat{\mathfrak{g}}$ were suggested in [FeP], by analogy with the finite-dimensional Gauss decomposition of G. However, in order to get a meaningful VOA structure - and the corresponding semi-infinite cohomology theory! - one must incorporate the twist by $\mathbf{w_0}$, built into the Bruhat decomposition.

One can recover the original Wakimoto realization from (2.18) by properly discarding the 'bar' variables. We use superscripts 'W' to distinguish the Wakimoto $\hat{\mathfrak{g}}_k$-action from (2.18).

Corollary 2.5. *The space* $\hat{W}_{\lambda,k} = \hat{F}(\beta,\gamma)\otimes\hat{F}_\varkappa(a)\otimes\mathbb{C}\mathbf{1}_\lambda$ *has the structure of a* $\hat{\mathfrak{g}}_k$*-module with* $k = \varkappa - h^\vee$*, defined by the formulas*

$$\begin{aligned}
&\mathbf{e}^W(z) = \gamma(z),\\
&\mathbf{h}^W(z) = 2:\beta(z)\gamma(z): + a(z),\\
&\mathbf{f}^W(z) = -:\beta(z)^2\gamma(z): - \beta(z)a(z) - k\,\beta'(z).
\end{aligned} \tag{2.20}$$

The $\hat{\mathfrak{g}}_k$*-module* $\hat{W}_{\lambda,k}$ *is called the Wakimoto module.*

Proof of Theorem 2.4. It suffices to show that modifying the standard Wakimoto actions by the extra terms

$$\begin{aligned}
\delta\mathbf{f}(z) &= \mathbf{f}(z) - \mathbf{f}^W(z) = \mathbb{Y}(-2,z)\bar{\gamma}(z),\\
\overline{\delta\mathbf{f}}(z) &= \bar{\mathbf{f}}(z) - \bar{\mathbf{f}}^W(z) = \mathbb{Y}(-2,z)\gamma(z),
\end{aligned}$$

does not destroy the operator product expansions.

We begin by showing that the commutation relations for $\hat{\mathfrak{g}}_k$ hold. Only those involving the modified quantum field $\mathbf{f}(z)$ need to be considered. We have:

$$\begin{aligned}
\mathbf{e}^W(z)\,\delta\mathbf{f}(w) &= \gamma(z)\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim 0,\\
\mathbf{h}^W(z)\,\delta\mathbf{f}(w) &= (2:\beta(z)\gamma(z): + a(z))\,\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim\\
&\sim a(z)\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim -\frac{2\mathbb{Y}(-2,w)}{z-w}\bar{\gamma}(w) = -\frac{2\,\delta\mathbf{f}(w)}{z-w},\\
\mathbf{f}^W(z)\,\delta\mathbf{f}(w) &= -\beta(z)a(z)\gamma(z)\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim \frac{2\,\mathbb{Y}(-2,w)}{z-w}\beta(w)\bar{\gamma}(w),\\
\delta\mathbf{f}(z)\,\delta\mathbf{f}(w) &= \mathbb{Y}(-2,z)\bar{\gamma}(z)\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim 0.
\end{aligned}$$

Using the operator product expansions above we immediately check that

$$\begin{aligned}
\mathbf{e}(z)\mathbf{f}(w) &= \mathbf{e}^W(z)\mathbf{f}^W(w) + \mathbf{e}^W(z)\,\delta\mathbf{f}(w) \sim \left(\frac{k}{(z-w)^2} + \frac{\mathbf{h}^W(w)}{z-w}\right) + 0 = \frac{k}{(z-w)^2} + \frac{\mathbf{h}(w)}{z-w},\\
\mathbf{h}(z)\mathbf{f}(w) &= \mathbf{h}^W(z)\mathbf{f}^W(w) + \mathbf{h}^W(z)\,\delta\mathbf{f}(w) \sim -\frac{2\,\mathbf{f}^W(w)}{z-w} + 0 = -\frac{2\,\mathbf{f}(w)}{z-w},\\
\mathbf{f}(z)\mathbf{f}(w) &= \mathbf{f}^W(z)\mathbf{f}^W(w) + \mathbf{f}^W(z)\,\delta\mathbf{f}(w) + \delta\mathbf{f}(z)\,\mathbf{f}^W(w) + \delta\mathbf{f}(z)\,\delta\mathbf{f}(w) \sim\\
&\sim 0 + \frac{2\,\mathbb{Y}(-2,w)}{z-w}\beta(w)\bar{\gamma}(w) + \frac{2\,\mathbb{Y}(-2,z)}{w-z}\beta(z)\bar{\gamma}(z) + 0 \sim 0,
\end{aligned}$$

and since the operator product expansions not involving $\mathbf{f}(z)$ are unchanged, we have proved the commutation relations for the (left) $\hat{\mathfrak{g}}_k$-action. Similarly, one verifies the commutation relations for the (right) $\hat{\mathfrak{g}}_{\bar{k}}$-action.

We now prove that the two actions of $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_{\bar{k}}$ commute. We have

$$\begin{aligned}
\bar{\mathbf{e}}^W(z)\,\delta\mathbf{f}(w) &= \bar{\gamma}(z)\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim 0,\\
\bar{\mathbf{h}}^W(z)\,\delta\mathbf{f}(w) &= \big(2 : \bar{\beta}(z)\bar{\gamma}(z) : + \bar{a}(z)\big)\,\mathbb{Y}(-2,w)\bar{\gamma}(w) \sim\\
&\sim 2\frac{\bar{\gamma}(z)}{z-w}\mathbb{Y}(-2,w) - \frac{2\mathbb{Y}(-2,w)}{z-w}\bar{\gamma}(w) \sim 0,
\end{aligned}$$

which implies that $\bar{\mathbf{e}}(z)\mathbf{f}(w) \sim \bar{\mathbf{h}}(z)\mathbf{f}(w) \sim 0$. Finally, we compute

$$\begin{aligned}
\bar{\mathbf{f}}^W(z)\,\delta\mathbf{f}(w) =&\Big(- : \bar{\beta}(z)^2\bar{\gamma}(z) : - \bar{k}\bar{\beta}'(z) - \bar{\beta}(z)\bar{a}(z)\Big)\Big(\mathbb{Y}(-2,w)\bar{\gamma}(w)\Big) \sim\\
&\sim -2\,\frac{\bar{\beta}(z)\bar{\gamma}(z)}{z-w}\mathbb{Y}(-2,w) + \frac{\bar{k}}{(z-w)^2}\mathbb{Y}(-2,w) -\\
&- \left(-\frac{2\mathbb{Y}(-2,w)}{z-w} : \bar{\beta}(z)\bar{\gamma}(w) : + \frac{: \bar{a}(w)\mathbb{Y}(-2,w) :}{z-w} - \frac{2\mathbb{Y}(-2,w)}{(z-w)^2}\right) \sim\\
&\sim \frac{(\bar{k}+2)\mathbb{Y}(-2,w)}{(z-w)^2} - \frac{: \bar{a}(w)\mathbb{Y}(-2,w) :}{z-w} = -\varkappa\frac{\mathbb{Y}(-2,w)}{(z-w)^2} - \frac{: \bar{a}(w)\mathbb{Y}(-2,w) :}{z-w}.
\end{aligned}$$

and similarly

$$\overline{\delta\mathbf{f}}(z)\mathbf{f}^W(w) \sim \varkappa\frac{\mathbb{Y}(-2,w)}{(z-w)^2} + \frac{: \bar{a}(w)\mathbb{Y}(-2,w) :}{z-w} \sim -\,\bar{\mathbf{f}}^W(z)\,\delta\mathbf{f}(w).$$

Therefore,

$$\bar{\mathbf{f}}(z)\mathbf{f}(w) = \bar{\mathbf{f}}^W(z)\mathbf{f}^W(w) + \bar{\mathbf{f}}^W(z)\,\delta\mathbf{f}(w) + \overline{\delta\mathbf{f}}(z)\,\mathbf{f}^W(w) + \overline{\delta\mathbf{f}}(z)\,\delta\mathbf{f}(w) \sim 0,$$

and we have established the commutativity of the two $\hat{\mathfrak{g}}$-actions.

Proposition 2.2 implies that $\hat{\mathbb{F}}_\varkappa$ is a vertex algebra. The formulas (2.18) can be written as

$$\begin{aligned}
\mathbf{e}(z) &= \mathcal{Y}(\gamma_{-1}\mathbf{1}_0, z),\\
\mathbf{h}(z) &= \mathcal{Y}(2\beta_0\gamma_{-1}\mathbf{1}_0 + a_{-1}\mathbf{1}_0, z),\\
\mathbf{f}(z) &= \mathcal{Y}(-(\beta_0)^2\gamma_{-1}\mathbf{1}_0 - a_{-1}\beta_0\mathbf{1}_0 - k\beta_{-1}\mathbf{1}_0 - \bar{\gamma}_{-1}\mathbf{1}_{-2}, z),
\end{aligned}$$

which means that the quantum fields (2.18) are special cases of the operators $\mathcal{Y}(\cdot, z)$. The same is true for the quantum fields (2.19). Therefore, the vertex algebra structure is compatible (in the vertex algebra sense) with the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module structure on $\hat{\mathbb{F}}_\varkappa$.

To give $\hat{\mathbb{F}}_\varkappa$ a VOA structure we need to introduce the Virasoro element. The Sugawara construction for the affine algebra $\hat{\mathfrak{g}}_k$ gives a Virasoro quantum field with central charge $c = \frac{3k}{k+h^\vee} = 3 - \frac{6}{\varkappa}$:

$$\begin{aligned}
L(z) &= \frac{1}{2\varkappa}\left(\frac{: \mathbf{h}^2(z) :}{2} + : \mathbf{e}(z)\mathbf{f}(z) : + : \mathbf{f}(z)\mathbf{e}(z) :\right) =\\
&= \frac{: a(z)^2 :}{4\varkappa} - \frac{a'(z)}{2\varkappa} - : \beta'(z)\gamma(z) : + \frac{1}{\varkappa}\,\mathbb{Y}(-2,z)\gamma(z)\bar{\gamma}(z).
\end{aligned} \tag{2.21}$$

We also note that

$$L(z) = L^W(z) - \frac{1}{\varkappa}\,\mathbb{Y}(-2,z)\gamma(z)\bar{\gamma}(z),$$

where $L^W(z)$ is the Virasoro quantum field given by the Sugawara construction for the standard Wakimoto realization (2.20).

Similarly, the affine algebra $\hat{\mathfrak{g}}_{\bar{k}}$ produces another Virasoro quantum field with central charge $\bar{c} = \frac{3\bar{k}}{\bar{k}+h^\vee} = 3 + \frac{6}{\varkappa}$:

$$\begin{aligned}\bar{L}(z) &= -\frac{1}{\varkappa}\left(\frac{:\bar{\mathbf{h}}^2(z):}{2} + :\bar{\mathbf{e}}(z)\bar{\mathbf{f}}(z): + :\bar{\mathbf{f}}(z)\bar{\mathbf{e}}(z):\right) = \\ &= -\frac{:\bar{a}(z)^2:}{4\varkappa} + \frac{\bar{a}'(z)}{2\varkappa} - :\bar{\beta}'(z)\bar{\gamma}(z): - \frac{1}{\varkappa}\,\mathbb{Y}(-2,z)\gamma(z)\bar{\gamma}(z).\end{aligned} \tag{2.22}$$

We set $\mathcal{L}(z) = L(z) + \bar{L}(z) = L^W(z) + \bar{L}^W(z)$. To show that $\mathcal{L}_{-1} = \mathcal{D}$, we check that

$$\mathcal{Y}(\mathcal{L}_{-1}v, z) = \frac{d}{dz}\mathcal{Y}(v,z), \quad v \in \hat{\mathbb{F}}_\varkappa, \tag{2.23}$$

which for all the generating quantum fields follows from straightforward computations.

Finally, the rank of the VOA $\hat{\mathbb{F}}_\varkappa$ is equal to

$$\operatorname{rank}\hat{\mathbb{F}}_\varkappa = c + \bar{c} = \left(3 - \frac{6}{\varkappa}\right) + \left(3 + \frac{6}{\varkappa}\right) = 6.$$

This concludes the proof of the theorem. □

2.4. **$\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module structure of $\hat{\mathbb{F}}_\varkappa$ for generic $\varkappa$.** We now prove the analogue of the Theorem 1.3, describing the structure of the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module $\hat{\mathbb{F}}_\varkappa$ for generic values of the parameter $\varkappa$.

For $\lambda \in \mathfrak{h}^*, k \in \mathbb{C}$ we denote by $\hat{V}_{\lambda,k}$ the irreducible $\hat{\mathfrak{g}}_k$-module, generated by a vector $\hat{v}$ satisfying $\mathfrak{g}_+\hat{v} = \mathfrak{n}_+\hat{v} = 0$ and $\mathbf{h}\hat{v} = \lambda\,\hat{v}$.

For any $\hat{\mathfrak{g}}_k$-module $\hat{V}$, the restricted dual space $\hat{V}'$ can be equipped with a $\hat{\mathfrak{g}}_k$-action by

$$\langle g_n\, v', v\rangle = -\langle v', \hat{\omega}(g_{-n})v\rangle, \qquad v \in \hat{V},\ v' \in \hat{V}',\ g \in \mathfrak{g},$$

where $\hat{\omega}$ is as in (2.1). We denote the resulting dual module by $\hat{V}^\star$.

An important source of $\hat{\mathfrak{g}}_k$-modules is the induced module construction. Any $\mathfrak{g}$-module V may be regarded as a module for the subalgebra $\mathfrak{p} = \bigoplus_{n\geq 0}\hat{\mathfrak{g}}[n]$, with $\mathfrak{g}[n]$ acting trivially for $n > 0$ and $\mathbf{k}$ acting as the multiplication by a scalar $k \in \mathbb{C}$. The induced $\hat{\mathfrak{g}}_k$-module $\hat{V}_k$ is defined as the space

$$\hat{V}_k = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{p})} V, \tag{2.24}$$

with $\hat{\mathfrak{g}}_k$ acting by left multiplication.

For the remainder of this section, we will assume that complex numbers $\varkappa, k, \bar{k}$ satisfy

$$\varkappa \notin \mathbb{Q}, \qquad k = \varkappa - h^\vee, \qquad \bar{k} = -\varkappa - h^\vee. \tag{2.25}$$

Theorem 2.6. *There exists a filtration*

$$0 \subset \hat{\mathbb{F}}^{(0)}_\varkappa \subset \hat{\mathbb{F}}^{(1)}_\varkappa \subset \hat{\mathbb{F}}^{(2)}_\varkappa = \hat{\mathbb{F}}_\varkappa \tag{2.26}$$

of $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodules of $\hat{\mathbb{F}}_\varkappa$ such that

$$\hat{\mathbb{F}}^{(2)}_\varkappa/\hat{\mathbb{F}}^{(1)}_\varkappa \cong \bigoplus_{\lambda\in\mathbf{P}^+} \hat{V}_{-\lambda-2,k} \otimes \hat{V}^\star_{-\lambda-2,\bar{k}}, \tag{2.27}$$

$$\hat{\mathbb{F}}^{(1)}_\varkappa/\hat{\mathbb{F}}^{(0)}_\varkappa \cong \bigoplus_{\lambda\in\mathbf{P}^+} \left(\hat{V}_{\lambda,k} \otimes \hat{V}^\star_{-\lambda-2,\bar{k}} \oplus \hat{V}_{-\lambda-2,k} \otimes \hat{V}^\star_{\lambda,\bar{k}}\right), \tag{2.28}$$

$$\hat{\mathbb{F}}^{(0)}_\varkappa \cong \bigoplus_{\lambda\in\mathbf{P}} \hat{V}_{\lambda,k} \otimes \hat{V}^\star_{\lambda,\bar{k}}. \tag{2.29}$$

Proof. The operator $\mathcal{L}_0$ determines a $\mathbb{Z}$-grading deg of $\hat{\mathbb{F}}_\varkappa$, which is explicitly described by

$$\deg \mathbf{1}_\lambda = 0, \qquad \deg X_n = -n \quad \text{ for } X = a, \bar{a}, \beta, \gamma, \bar{\beta}, \bar{\gamma}. \tag{2.30}$$

The lowest graded subspace $\hat{\mathbb{F}}_\varkappa[0] = F(\beta_0, \gamma_0) \otimes F(\bar{\beta}_0, \bar{\gamma}_0) \otimes \mathbb{C}[\mathbf{P}]$ of the vertex algebra $\hat{\mathbb{F}}_\varkappa$ is identified with the Fock space $\mathbb{F}$ for the finite-dimensional Lie algebra $\mathfrak{g}$. Moreover, since $\varkappa$ is generic, $\hat{\mathbb{F}}_\varkappa$ can be constructed as the induced $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module from the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathbb{F}$:

$$\hat{\mathbb{F}}_\varkappa = \mathcal{U}(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})} \mathbb{F}.$$

We construct the filtration (2.26) by inducing it from the finite-dimensional one (1.10):

$$\hat{\mathbb{F}}_\varkappa^{(0)} = \mathcal{U}(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})} \mathbb{F}^{(0)}, \qquad \hat{\mathbb{F}}_\varkappa^{(1)} = \mathcal{U}(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})} \mathbb{F}^{(1)}.$$

It is easy to check that (1.11),(1.12),(1.13) respectively imply (2.27),(2.28),(2.29), which proves the theorem. □

The analogue of the Corollary 1.4, describing the realization of the subalgebra $\mathfrak{R}(G) \subset \mathfrak{R}(G_0)$, is given below.

Theorem 2.7. *There exists a subspace $\hat{\mathbf{F}}_\varkappa \subset \hat{\mathbb{F}}_\varkappa$, satisfying*

(1) *$\hat{\mathbf{F}}_\varkappa$ is a vertex operator subalgebra of $\hat{\mathbb{F}}_\varkappa$, and is generated by the quantum fields* (2.18), (2.19) *and $\mathbb{Y}(1, z)$. In particular, $\mathbf{F}$ is a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodule of $\hat{\mathbb{F}}_\varkappa$.*
(2) *As a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module, $\hat{\mathbf{F}}_\varkappa$ is generated by the vectors $\{\mathbf{1}_\lambda\}_{\lambda\in\mathbf{P}^+}$, and we have*

$$\hat{\mathbf{F}}_\varkappa \cong \bigoplus_{\lambda\in\mathbf{P}^+} \hat{V}_{\lambda,k} \otimes \hat{V}^\star_{\lambda,\bar{k}}. \tag{2.31}$$

Proof. As before, we identify the lowest graded subspace $\hat{\mathbb{F}}_\varkappa[0] \subset \hat{\mathbb{F}}_\varkappa$ with the Fock space $\mathbb{F}$ for the finite-dimensional Lie algebra $\mathfrak{g}$. Recall from Corollary 1.4 that the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathbb{F}$ contains the distinguished submodule $\mathbf{F}$. We define the subspace $\hat{\mathbf{F}}_\varkappa$ as the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodule of $\hat{\mathbb{F}}_\varkappa$, induced from $\mathbf{F}$:

$$\hat{\mathbf{F}}_\varkappa = \mathcal{U}(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})} \mathbf{F}.$$

It immediately follows from Corollary 1.4 that $\hat{\mathbf{F}}_\varkappa$ is generated by the vectors $\{\mathbf{1}_\lambda\}_{\lambda\in\mathbf{P}^+}$, and has the decomposition (2.31).

Next, we need to show that $\hat{\mathbf{F}}_\varkappa$ is a vertex subalgebra. Let $\hat{\mathbf{F}}'_\varkappa$ denote the space, spanned by the Laurent coefficients of $\hat{\mathbb{F}}_\varkappa$-valued fields $\mathcal{Y}(a, z)b$ for all possible $a, b \in \hat{\mathbf{F}}_\varkappa$. We will establish that $\hat{\mathbf{F}}'_\varkappa = \hat{\mathbf{F}}_\varkappa$.

Indeed, $\hat{\mathbf{F}}'_\varkappa$ is a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodule of $\hat{\mathbb{F}}_\varkappa$, and can be induced from its lowest graded component $\mathbf{F}' = \hat{\mathbf{F}}'_\varkappa[0]$, which is a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathbb{F}$. It suffices to prove that $\mathbf{F}' = \mathbf{F}$.

For any $a, b \in \mathbf{F}$, the lowest graded component of $\mathcal{Y}(a, z)b$ is equal to the product ab in the algebra $\mathbb{F}$, and since $\mathbf{F}$ is a subalgebra, we have $ab \in \mathbf{F}$. (Note that any element $a \in \mathbf{F}$ can be obtained this way, for example, by taking $b = \mathbf{1}$). Using the commutation relations with the two copies of $\hat{\mathfrak{g}}$, we can prove that the lowest graded component of $\mathcal{Y}(a, z)b$ lies in $\mathbf{F}$ for any $a, b \in \hat{\mathbf{F}}_\varkappa$.

It follows that $\mathbf{F}' = \mathbf{F}$ and hence $\hat{\mathbf{F}}'_\varkappa = \hat{\mathbf{F}}_\varkappa$, which means that the restrictions of the operators $\mathcal{Y}(\cdot, z)$, corresponding to the subspace $\hat{\mathbf{F}}_\varkappa$, are well-defined. Thus $\hat{\mathbf{F}}_\varkappa$ is a vertex subalgebra of $\hat{\mathbb{F}}_\varkappa$. It is clear that as a vertex subalgebra $\hat{\mathbf{F}}_\varkappa$ is generated by the quantum

fields (2.18), (2.19) and $\{\mathbb{Y}(\lambda, z)\}_{\lambda\in\mathbf{P}^+}$, and the latter are generated by the single operator $\mathbb{Y}(1, z)$.

Finally, $\hat{\mathbf{F}}_\varkappa$ contains both $L^W(z)$ and $\bar{L}^W(z)$ - hence also $\mathcal{L}(z)$ - and therefore is a vertex operator subalgebra of $\hat{\mathbb{F}}_\varkappa$. □

Remark 6. The vertex operator algebras $\hat{\mathbf{F}}_\varkappa$ and $\hat{\mathbb{F}}_\varkappa$ give explicit realizations of the modified regular representations $\mathfrak{R}'_\varkappa(\hat{G})$ and $\mathfrak{R}'_\varkappa(\hat{G}_0)$ we discussed in the introduction. It would be interesting to construct them invariantly by using the correlation functions approach [FZ], interpreting the rational functions $\langle \mathbf{1}', \mathcal{Y}(v_1, z_1)\dots\mathcal{Y}(v_n, z_n)\mathbf{1}\rangle$ for $v_1, \dots, v_n \in \mathbb{F} \cong \hat{\mathbb{F}}_\varkappa[0]$ as solutions of differential equations similar to the Knizhnik-Zamolodchikov equations.

2.5. **Semi-infinite cohomology of $\hat{\mathfrak{g}}$.** The fact that the level of the diagonal action of $\hat{\mathfrak{g}}$ in the modified regular representations is equal to the special value $-2h^\vee$ allows us to introduce the semi-infinite cohomology of $\hat{\mathfrak{g}}$ with coefficients in $\hat{\mathbb{F}}_\varkappa$ and in $\hat{\mathbf{F}}_\varkappa$. In this section we show that for generic values of $\varkappa$ these cohomologies lead to the same algebras of formal characters as in the finite-dimensional case.

We recall the definition of the semi-infinite cohomology [Fe, FGZ]. The main new ingredient is the "space of semi-infinite forms" $\hat{\mathbf{\Lambda}}^{\frac{\infty}{2}}$, which replaces the finite-dimensional exterior algebra $\mathbf{\Lambda}$. We summarize its properties in the following

Proposition 2.8. *Let* $\hat{\mathbf{\Lambda}}^{\frac{\infty}{2}} = \bigwedge \hat{\mathfrak{g}}_- \otimes \bigwedge(\hat{\mathfrak{g}}'_+ \oplus \mathfrak{g}')$. *Then*

(1) *The Clifford algebra, generated by* $\{\iota(g_n), \varepsilon(g'_n)\}_{g\in\mathfrak{g}, g'\in\mathfrak{g}', n\in\mathbb{Z}}$ *with relations*
$$\{\iota(x_m), \iota(y_n)\} = \{\varepsilon(x'_m), \varepsilon(y'_n)\} = 0, \qquad \{\iota(x_m), \varepsilon(y'_n)\} = \delta_{m,n}\,\langle y', x\rangle. \tag{2.32}$$
acts irreducibly on $\hat{\mathbf{\Lambda}}^{\frac{\infty}{2}}$, *so that for any* $\omega_- \in \bigwedge\hat{\mathfrak{g}}_-$, $\omega_+ \in \bigwedge(\hat{\mathfrak{g}}'_+ \oplus \mathfrak{g}')$ *we have*
$$\iota(x_n)(\omega_-\otimes 1) = \begin{cases} 0, & n\ge 0 \\ (x_n\wedge\omega_-)\otimes 1, & n<0\end{cases}, \qquad \varepsilon(x'_n)(1\otimes\omega_+) = \begin{cases} 1\otimes(x'_n\wedge\omega_+), & n\ge 0\\ 0, & n<0\end{cases}.$$

(2) $\hat{\mathbf{\Lambda}}^{\frac{\infty}{2}}$ *is a bi-graded vertex superalgebra, with vacuum* $\mathbf{1} = 1\otimes 1$, *and generated by*
$$\iota(x,z) = \sum_{n\in\mathbb{Z}} \iota(x_n) z^{-n-1}, \qquad |\iota(x,z)| = -1, \quad \deg\iota(x,z) = 1, \quad x\in\mathfrak{g},$$
$$\varepsilon(x',z) = \sum_{n\in\mathbb{Z}} \varepsilon(x'_{-n}) z^{-n}, \qquad |\varepsilon(x',z)| = 1, \qquad \deg\varepsilon(x',z) = 0, \quad x'\in\mathfrak{g}'.$$

(3) $\hat{\mathbf{\Lambda}}^{\frac{\infty}{2}}$ *has a* $\hat{\mathfrak{g}}$*-module structure on the level* $\mathbf{k} = 2h^\vee$, *defined by*
$$\pi(x_n) = \sum_{m\in\mathbb{Z}}\sum_i :\varepsilon((g'_i)_m)\iota([g_i, x]_{n+m}):, \qquad x\in\mathfrak{g}.$$

One can think of $\bigwedge\hat{\mathfrak{g}}_-$ as the space spanned by formal "semi-infinite" forms
$$\omega = \xi'_{i_1}\wedge\xi'_{i_2}\wedge\xi'_{i_3}\wedge\dots, \qquad i_{n+1} = i_n + 1 \text{ for } n \gg 0,$$
where $\{\xi_j\}_{j\in\mathbb{N}}$ is a homogeneous basis of $\hat{\mathfrak{g}}_-$. A monomial $\xi_{j_1}\wedge\dots\wedge\xi_{j_m}\in\bigwedge\hat{\mathfrak{g}}_-$ is identified with the semi-infinite form with the corresponding factors missing:
$$\omega = \pm\,\xi'_1\wedge\xi'_2\wedge\dots\wedge\xi'_{j_1-1}\wedge\widehat{\xi'_{j_1}}\wedge\xi'_{j_1+1}\wedge\dots\wedge\xi_{j_m-1}\wedge\widehat{\xi_{j_m}}\wedge\xi_{j_m+1}\wedge\dots.$$
In other words, if $\xi_j\in\hat{\mathfrak{g}}$, then $\iota(\xi_j)$ operates as usual by eliminating the factor ξ'_j.

Definition 3. The BRST complex, associated with a $\hat{\mathfrak{g}}$-module $\hat{V}$ on the level $k = -2h^\vee$, is the complex $C^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V}) = \hat{\mathbf{\Lambda}}^{\frac{\infty}{2}+\bullet} \otimes \hat{V}$, with the differential

$$\hat{\mathbf{d}} = \sum_{n\in\mathbb{Z}} \sum_i \varepsilon((g_i')_n)\pi_{\hat{V}}((g_i)_n) - \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \sum_{i,j} :\varepsilon((g_i)'_m)\varepsilon((g_j)'_n)\iota([g_i, g_j]_{m+n}):, \tag{2.33}$$

where $\{g_i\}$ is any basis of $\mathfrak{g}$, and $\{g_i'\}$ is the dual basis of $\mathfrak{g}'$. The corresponding cohomology is denoted $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})$.

The BRST complex above gives the relative (to the center) version of the semi-infinite cohomology. We don't consider any other type of cohomology, and thus simply drop the word 'relative' everywhere. The condition $k = -2h^\vee$ is equivalent to $\hat{\mathbf{d}}^2 = 0$.

If $\hat{V}$ is a vertex algebra, then its semi-infinite cohomology inherits a vertex superalgebra structure [LZ1].

The following theorem is similar to the reduction theorem of [FGZ] (see also [L]), and relates the semi-infinite cohomology for generic values of $\varkappa$ with the classical cohomology of Lie algebras.

Theorem 2.9. *Let V be a $\mathfrak{g} \oplus \mathfrak{g}$-module, and let $\varkappa \in \mathbb{C}$ be generic. Set $k = \varkappa - h^\vee$ and $\bar{k} = -\varkappa - h^\vee$, and denote $\hat{V}$ be the induced $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-module. Then with respect to the diagonal $\mathfrak{g}$-action $\hat{V}$ is a level $\mathbf{k} = -2h^\vee$ module, and*

$$H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V}) \cong H^\bullet(\mathfrak{g}, V).$$

Proof. As a vector space, the module $\hat{V}$ has a decomposition

$$\hat{V} = \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes V \otimes \mathcal{U}(\hat{\mathfrak{g}}_+)',$$

where we identified the factor $\mathcal{U}(\hat{\mathfrak{g}}_-)$, coming from the right induced action of $\hat{\mathfrak{g}}_{\bar{k}}$, with $\mathcal{U}(\hat{\mathfrak{g}}_+)'$ using the non-degenerate (since $\varkappa$ is generic!) contravariant pairing. Therefore, as vector spaces

$$C^\bullet(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V}) = C^\bullet_- \otimes C^\bullet_0 \otimes C^\bullet_+, \tag{2.34}$$

where

$$C^\bullet_- = \bigwedge \hat{\mathfrak{g}}_- \otimes \mathcal{U}(\hat{\mathfrak{g}}_-), \qquad C^\bullet_0 = \bigwedge \mathfrak{g}' \otimes V, \qquad C^\bullet_+ = \bigwedge \hat{\mathfrak{g}}'_+ \otimes \mathcal{U}(\hat{\mathfrak{g}}_+)^*$$

We write the differential $\hat{\mathbf{d}}$ as

$$\hat{\mathbf{d}} = \mathbf{d}_- + \mathbf{d}_0 + \mathbf{d}_+ + \boldsymbol{\delta},$$

where $\mathbf{d}_\pm$ are the BRST differentials for $\hat{\mathfrak{g}}_\pm$,

$$\mathbf{d}_- = \sum_{n<0} \sum_i \varepsilon((g_i')_n)\pi_l((g_i)_n) - \frac{1}{2} \sum_{m,n<0} \sum_{i,j} :\varepsilon((g_i)'_m)\varepsilon((g_j)'_n)\iota([g_i, g_j]_{m+n}):,$$

$$\mathbf{d}_+ = \sum_{n>0} \sum_i \varepsilon((g_i')_n)\pi_r((g_i)_n) - \frac{1}{2} \sum_{m,n>0} \sum_{i,j} :\varepsilon((g_i)'_m)\varepsilon((g_j)'_n)\iota([g_i, g_j]_{m+n}):,$$

the differential $\mathbf{d}_0$ is defined as in (1.24) with the $\mathfrak{g}$-action π_V replaced by

$$\pi(x) = \pi_{\hat{V}}(x) + \sum_{n\neq 0} :\varepsilon((g_j)'_n)\iota([x, g_j]_n):, \qquad x \in \mathfrak{g}, \tag{2.35}$$

and $\boldsymbol{\delta}$ includes all the remaining terms:

$$\boldsymbol{\delta} = \sum_{n>0}\sum_{i} \varepsilon((g_i')_n)\pi_l((g_i)_n) + \sum_{n<0}\sum_{i} \varepsilon((g_i')_n)\pi_r((g_i)_n) - \\ - \sum_{m>0,n<0}\sum_{i,j} : \varepsilon((g_i)'_m)\varepsilon((g_j)'_n)\iota([g_i, g_j]_{m+n}) : .$$

Following [FGZ], we introduce the skewed degree $f\deg$ by

$$f\deg(w_- \otimes w_0 \otimes w_+) = \deg w_+ - \deg w_-, \qquad w_\pm \in C_\pm, \quad w_0 \in C_0,$$

where the 'deg' gradings in the complexes $C_\pm$ are inherited from $C^{\frac{\infty}{2}}(\hat{\mathfrak{g}}, \mathbf{k}; \hat{V})$. We set

$$\mathfrak{B}^p = \left\{ v \in C^{\frac{\infty}{2}}(\hat{\mathfrak{g}}, \mathbf{k}; \hat{V}) \;\middle|\; f\deg v \geq p \right\}.$$

One can check that $\mathbf{d}_\pm$ and $\mathbf{d}_0$ preserve the filtered degree, and that $\boldsymbol{\delta}(\mathfrak{B}^p) \subset \mathfrak{B}^{p+1}$. Thus, $\{\mathfrak{B}^p\}_{p\in\mathbb{Z}}$ is a decreasing filtration of the complex $C^{\frac{\infty}{2}}(\hat{\mathfrak{g}}, \mathbf{k}; \hat{V})$, and the associated graded complex has the reduced differential

$$\mathbf{d}_{red} = \mathbf{d}_- + \mathbf{d}_0 + \mathbf{d}_+.$$

We now compute the corresponding reduced cohomology, which will provide a bridge to $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})$.

It is clear that $\mathbf{d}_\pm^2 = \{\mathbf{d}_+, \mathbf{d}_-\} = 0$, and that the differentials $\mathbf{d}_\pm : C_\pm^\bullet \to C_\pm^{\bullet+1}$ act in their respective factors of (2.34). One can also check that $(\mathbf{d}_0)^2 = \{\mathbf{d}_0, \mathbf{d}_\pm\} = 0$; moreover,

$$\mathbf{d}_0(C_-^\bullet \otimes C_0^\bullet \otimes C_+^\bullet) \subset (C_-^\bullet \otimes C_0^{\bullet+1} \otimes C_+^\bullet)$$

despite the fact that $\mathbf{d}_0$ does not act in $C_0^\bullet$.

It is a well-known fact in homological algebra that

$$H^n(C_+, \mathbf{d}_+) = H^n(\hat{\mathfrak{g}}_+; \mathcal{U}(\hat{\mathfrak{g}}_+)') = \delta_{n,0}\,\mathbb{C},$$

with $1 \otimes 1' \in C_+$ representing the non-trivial cohomology class. Similarly, one has

$$H^n(C_-, \mathbf{d}_-) = H_{-n}(\hat{\mathfrak{g}}_-; \mathcal{U}(\hat{\mathfrak{g}}_-)) = \delta_{n,0}\,\mathbb{C},$$

and $1 \otimes 1 \in C_-$ represents the non-trivial cohomology. Further, one can check that the subspace $1 \otimes C_0^\bullet \otimes 1 \subset C^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})$ is stabilized by $\mathbf{d}_0$, and that the $\mathfrak{g}$-action (2.35) on that subspace reduces to the $\mathfrak{g}$ action $1 \otimes \pi_V \otimes 1$. It follows that

$$H^\bullet_{red}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V}) \cong H^\bullet(1 \otimes C_0 \otimes 1, \mathbf{d}_0) \cong H^\bullet(C_0, \mathbf{d}) \cong H^\bullet(\mathfrak{g}, V).$$

We now return to the cohomology of $C^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})$. Since $\hat{\mathbf{d}}$ preserves the 'deg' grading, it can be computed separately for each subcomplex $C^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})[m], m \in \mathbb{Z}$. The filtration $\{\mathfrak{B}^p[m]\}_{p\in\mathbb{Z}}$ of this complex is finite for each m, and leads to a finitely converging spectral sequence with $E_1^{p,q}[m] = H^q_{red}(\mathfrak{B}^p[m]/\mathfrak{B}^{p+1}[m])$.

For $m \neq 0$ we have $H^q_{red}(\mathfrak{B}^p[m]/\mathfrak{B}^{p+1}[m]) = 0$ for all p, hence the spectral sequence is zero, and $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})[m] = 0$. For $m = 0$ we note that

$$\mathfrak{B}^0[0] = C^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})[0], \qquad \mathfrak{B}^1[0] = 0,$$

which means that $E_1^{p,q}[0] = 0$ unless $p = 0$, and the collapsing spectral sequence implies

$$H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{V})[0] \cong H^\bullet_{red}(\mathfrak{B}^0[0]/\mathfrak{B}^1[0]) \cong H^\bullet(\mathfrak{g}; V).$$

This completes the proof of the theorem. $\square$

Corollary 2.10. *The vertex superalgebras* $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbf{F}}_\varkappa)$, $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_\varkappa)$ *degenerate into commutative superalgebras. Moreover, we have commutative superalgebra isomorphisms*

$$H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbf{F}}_\varkappa) \cong H^\bullet(\mathfrak{g}; \mathbf{F}), \qquad H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_\varkappa) \cong H^\bullet(\mathfrak{g}; \mathbb{F}). \tag{2.36}$$

In particular,

$$H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbf{F}}_\varkappa) \cong H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_\varkappa) \cong \mathbb{C}[\mathbf{P}]^W.$$

Proof. Theorem 2.9 gives us isomorphisms (2.36) on the level of vector spaces. It is also clear from its proof that the semi-infinite cohomology is concentrated in the subspace of $\deg = 0$, and thus the operators $\mathcal{Y}(\cdot, z)$ on cohomology are reduced to their constant terms. In particular, they are independent of z, which means that the vertex superalgebra degenerates into a commutative superalgebra. The multiplication is easily traced back to the multiplications in $\mathbf{F} \cong \hat{\mathbf{F}}_\varkappa[0]$ and in the exterior algebra $\mathbf{\Lambda} = \bigwedge \mathfrak{g}'$, which shows that (2.36) are superalgebra isomorphisms. ☐

3. Modified regular representations of the Virasoro algebra.

3.1. **Virasoro algebra and the quantum Drinfeld-Sokolov reduction.** In this section we present a construction of the regular representation of the Virasoro algebra, which goes in parallel with constructions in the previous sections. However, instead of beginning with a space of functions on the corresponding group (which is, strictly speaking, a semigroup in the complex case), we will use the quantum Drinfeld-Sokolov reduction [FeFr2] (see also [FrB] and references therein), applied to the modified regular representations of $\hat{\mathfrak{g}}$ constructed in Section 1. As a result we obtain certain bimodules over the Virasoro algebra, which have the structure similar to their affine counterparts. The result of the quantum Drinfeld-Sokolov reduction applied to the actual regular representation of $\hat{\mathfrak{g}}$ should have a standard interpretation in terms of the space of functions on the Virasoro semigroup, but we will not need this fact for our purposes.

Recall that the Virasoro algebra Vir is the infinite-dimensional complex Lie algebra, generated by $\{L_n\}_{n\in\mathbb{Z}}$ and a central element $\mathbf{c}$, subject to the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\,\mathbf{c}.$$

The Virasoro algebra has a $\mathbb{Z}$-grading $\mathrm{Vir} = \oplus_{n\in\mathbb{Z}} \mathrm{Vir}[n]$, determined by

$$\deg L_n = -n, \qquad \deg \mathbf{c} = 0.$$

There is a functorial correspondence between certain representations of affine Lie algebras and their $\mathcal{W}$-algebra counterparts, called the quantum Drinfeld-Sokolov reduction [FeFr2] (see also [FrB] and references therein). We review this procedure for the case $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2, \mathbb{C})$, when the corresponding $\mathcal{W}$-algebra is identified with the Virasoro algebra.

Definition 4. For any $\hat{\mathfrak{g}}_k$-module $\hat{V}$, the complex $(C_{DS}(\hat{V}), \mathbf{d}_{DS})$,

$$C_{DS}(\hat{V}) = \hat{V} \otimes \hat{\Lambda}(\psi, \psi^*), \qquad \mathbf{d}_{DS} = \sum_{n\in\mathbb{Z}} \psi_n^* \pi_{\hat{V}}(\mathbf{e}_n) + \psi_1^*,$$

is called the BRST complex of the quantum Drindeld-Sokolov reduction. The corresponding cohomology is denoted $H_{DS}(\hat{V})$.

The BRST complex above is very similar to the semi-infinite cohomology complex for the nilpotent loop algebra $\hat{\mathfrak{n}}_+ = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\mathbf{e}_n$. Indeed, the corresponding space of semi-infinite forms $\mathbf{\Lambda}^{\frac{\infty}{2}}(\hat{\mathfrak{n}}_+)$ is identified with $\hat{\Lambda}(\psi,\psi^*)$ by $\iota(\mathbf{e}_n) \equiv \psi_n,\ \varepsilon(\mathbf{e}'_n) \equiv \psi^*_{-n}$, and the only modification is the additional term ψ^*_1 in the differential.

The BRST complex inherits the gradings $|\cdot|$ and deg from $\hat{\Lambda}(\psi,\psi^*)$ and $\hat{V}$. Since $|\mathbf{d}_{DS}| = 1$, the grading $|\cdot|$ descends to the cohomology $H_{DS}(\hat{V})$. However, with respect to the other grading, the differential $\mathbf{d}_{DS}$ is not homogeneous. We introduce a modified grading deg′ by

$$\deg' \mathbf{e}(z) = 0, \qquad \deg' \mathbf{h}(z) = 1, \qquad \deg' \mathbf{f}(z) = 2,$$
$$\deg' \psi(z) = 0, \qquad \deg' \psi^*(z) = 1.$$

The differential $\mathbf{d}_{DS}$ then satisfies $\deg' \mathbf{d}_{DS} = 0$, and the grading deg′ descends to $H_{DS}(\hat{V})$.

The cohomology $H^0_{DS}(\hat{\mathfrak{g}}_k)$ of the vacuum module inherits a vertex algebra structure. We have the following result (details of the proof can be found in [FrB]).

Proposition 3.1. *For $k \neq -h^\vee$ we have $H^0_{DS}(\hat{\mathfrak{g}}_k) \cong \mathrm{Vir}_c$, where $c = 1 - \frac{6}{k+h^\vee} - 6k$.*

For any $\hat{\mathfrak{g}}_k$-module $\hat{V}$, the vertex algebra $\mathrm{Vir}_c \cong H^0_{DS}(\hat{\mathfrak{g}}_k)$ acts on $H^0_{DS}(\hat{V})$. For $\varkappa \neq 0$, set $\tilde{F}_{\lambda,\varkappa} = H^0_{DS}(\hat{W}_{\lambda,\varkappa-h^\vee})$. The following identifies the Vir_c-module structure on $\tilde{F}_{\lambda,\varkappa}$.

Proposition 3.2. *Let $\varkappa \neq 0$, and let $c = 13 - 6\varkappa - \frac{6}{\varkappa}$. Then $\tilde{F}_{\lambda,\varkappa} \cong \hat{F}_\varkappa(a) \otimes \mathbb{C}\mathbf{1}_\lambda$ as a vector space, and the Vir_c-action is given by*

$$L^F(z) = \frac{1}{4\varkappa} : a(z)^2 : + \frac{\varkappa - 1}{2\varkappa}\, a'(z). \tag{3.1}$$

Proof. In the vector space factorization of $\hat{W}_{\lambda,\varkappa-h^\vee} = \hat{F}(\beta,\gamma) \otimes \hat{F}_\varkappa(a) \otimes \mathbb{C}\mathbf{1}_\lambda$, the differential $\mathbf{d}_{DS}$ acts only in the first component. Therefore, we must have

$$\tilde{F}_{\lambda,\varkappa} = H_{DS}(\hat{F}(\beta,\gamma)) \otimes \hat{F}_\varkappa(a) \otimes \mathbb{C}\mathbf{1}_\lambda.$$

A spectral sequence reduces the cohomology $H^0_{DS}(\hat{F}(\beta,\gamma))$ to the cohomology of the semi-infinite Weil complex $\hat{F}(\beta,\gamma) \otimes \hat{\Lambda}(\psi,\psi^*)$. The latter splits into an infinite product of finite-dimensional Weil complexes, and thus has one-dimensional cohomology, concentrated in degree 0.

The inclusion of vertex algebras $\hat{\mathfrak{g}}_{\varkappa-h^\vee} \hookrightarrow \hat{W}_{0,\varkappa-h^\vee}$ induces an inclusion $\mathrm{Vir}_c \hookrightarrow \tilde{F}_{0,\varkappa}$, and the explicit formula (3.1) for $L(z)$ in terms of $a(z)$ is a result of a direct computation. □

The realization (3.1) of Virasoro modules was known long before the quantum Drinfeld-Sokolov reduction, and is called the Feigin-Fuks construction in the literature. We use the superscript "F" to distinguish this standard action from the modified Virasoro actions, which we will be considering later.

3.2. **Bosonic realization of the regular representation.** The Virasoro analogue of the Peter-Weyl theorem is more subtle than in the case of classical and affine Lie algebras. There is no clear way to calculate the two commuting Vir-actions in a way similar to Theorem 1.2 and Theorem 2.1. However, there exists a Fock space realization analogous to Theorem 2.4, which we will call the regular representation of the Virasoro algebra.

Theorem 3.3. *Let $\varkappa \neq 0$, and let $c = 13 - 6\varkappa - \frac{6}{\varkappa}$ and $\bar{c} = 13 + 6\varkappa + \frac{6}{\varkappa}$.*

(1) *The space* $\tilde{\mathbb{F}}_\varkappa$ *has a* $\mathrm{Vir}_c \oplus \mathrm{Vir}_{\bar{c}}$*-module structure, defined by*

$$L(z) = \frac{1}{4\varkappa} : a(z)^2 : + \frac{\varkappa - 1}{2\varkappa} a'(z) - \frac{1}{\varkappa}\, \mathbb{Y}(-2, z), \tag{3.2}$$

$$\bar{L}(z) = -\frac{1}{4\varkappa} : \bar{a}(z)^2 : + \frac{\varkappa + 1}{2\varkappa} \bar{a}'(z) + \frac{1}{\varkappa}\, \mathbb{Y}(-2, z). \tag{3.3}$$

(2) *The space* $\tilde{\mathbb{F}}_\varkappa$ *has a compatible VOA structure with* $\operatorname{rank} \tilde{\mathbb{F}}_\varkappa = 26$*.*

Proof. The formulas (3.2),(3.3) are nothing else but the result of the two-sided quantum Drinfeld-Sokolov reduction, which consists of two reductions applied separately to the two commuting $\hat{\mathfrak{g}}$-actions of Theorem 2.4, cf. formulas (2.21),(2.22) and Proposition 3.2.

Rather than give detailed proof of this fact, we choose to verify the commutation relations directly. Introduce notation

$$\delta L(z) = L(z) - L^F(z) = \frac{1}{\varkappa}\, \mathbb{Y}(-2, z),$$
$$\overline{\delta L}(z) = \bar{L}(z) - \bar{L}^F(z) = -\frac{1}{\varkappa}\, \mathbb{Y}(-2, z).$$

Without the additional terms $\delta L(z), \overline{\delta L}(z)$, both (3.2) and (3.3) give two commuting copies of the standard construction (3.1) with the specified central charges. Therefore, it suffices to show that the presence of these extra terms does not violate the commutation relations for $\mathrm{Vir}_c \oplus \mathrm{Vir}_{\bar{c}}$.

Straightforward computations immediately show that

$$\delta L(z)\, \delta L(w) \sim \delta L(z)\, \overline{\delta L}(w) \sim \overline{\delta L}(z)\, \overline{\delta L}(w) \sim 0.$$

$$L^F(z)\mathbb{Y}(-2, w) \sim \frac{\mathbb{Y}(-2, w)}{(z - w)^2} - \frac{1}{\varkappa} \frac{: a(w)\mathbb{Y}(-2, w) :}{z - w},$$

$$\bar{L}^F(z)\mathbb{Y}(-2, w) \sim \frac{\mathbb{Y}(-2, w)}{(z - w)^2} + \frac{1}{\varkappa} \frac{: \bar{a}(w)\mathbb{Y}(-2, w) :}{z - w}.$$

We now prove the commutation relations for the action (3.2). We have

$$L(z)L(w) - L^F(z)L^F(w) = L^F(z)\, \delta L(w) + \delta L(z) L^F(w) + \delta L(z)\, \delta L(w) \sim$$
$$\sim \frac{1}{\varkappa}\left(\frac{\mathbb{Y}(-2, w)}{(z - w)^2} - \frac{1}{\varkappa} \frac{: a(w)\mathbb{Y}(-2, w) :}{z - w}\right) + \frac{1}{\varkappa}\left(\frac{\mathbb{Y}(-2, z)}{(z - w)^2} + \frac{1}{\varkappa} \frac{: a(z)\mathbb{Y}(-2, w) :}{z - w}\right) \sim$$
$$\sim \frac{1}{\varkappa}\left(\frac{2\, \mathbb{Y}(-2, w)}{(z - w)^2} + \frac{\mathbb{Y}'(-2, w)}{z - w}\right) \sim \frac{2\, \delta L(w)}{(z - w)^2} + \frac{(\delta L)'(w)}{z - w},$$

and therefore

$$L(z)L(w) \sim L^F(z)L^F(w) + \frac{2\, \delta L(w)}{(z - w)^2} + \frac{(\delta L)'(w)}{z - w} \sim \left(\frac{c/2}{(z - w)^4} + \frac{2L^F(w)}{(z - w)^2} + \frac{(L^F)'(w)}{z - w}\right) +$$
$$+ \frac{2\, \delta L(w)}{(z - w)^2} + \frac{(\delta L)'(w)}{(z - w)^2} = \frac{c/2}{(z - w)^4} + \frac{2L(w)}{(z - w)^2} + \frac{L'(w)}{z - w}.$$

We have established that adding the extra term $\delta L(z)$ to the action (3.1) preserves the commutation relations for Vir_c. Similarly, the formula (3.3) gives a representation of $\mathrm{Vir}_{\bar{c}}$.

We now show that the two actions of Vir_c and $\mathrm{Vir}_{\bar c}$ commute. Using (2.16), we get

$$\begin{aligned}\delta L(z)\bar L^F(w) &\sim \frac{1}{\varkappa}\left(\frac{\mathbb{Y}(-2,z)}{(z-w)^2}-\frac{1}{\varkappa}\frac{:\bar a(z)\mathbb{Y}(-2,z):}{z-w}\right)\sim\\ &\sim \frac{1}{\varkappa}\left(\frac{\mathbb{Y}(-2,w)}{(z-w)^2}+\frac{\mathbb{Y}'(-2,w)}{z-w}-\frac{1}{\varkappa}\frac{:\bar a(w)\mathbb{Y}(-2,w):}{z-w}\right)\sim\\ &\sim \frac{1}{\varkappa}\left(\frac{\mathbb{Y}(-2,w)}{(z-w)^2}-\frac{1}{\varkappa}\frac{:a(w)\mathbb{Y}(-2,w):}{z-w}\right).\end{aligned}$$

Note that (2.16) implies $\delta L(z)\bar L^F(w)\sim -L^F(z)\,\overline{\delta L}(w)$, and thus

$$L(z)\bar L(w)=L^F(z)\bar L^F(w)+L^F(z)\,\overline{\delta L}(w)+\delta L(z)\bar L^F(w)+\delta L(z)\overline{\delta L}(w)\sim 0,$$

which means that the two Virasoro actions commute.

It is easy to see that the formula (3.2) can be written as

$$L(z)=\mathcal{Y}\left(\frac{(a_{-1})^2}{4\varkappa}\mathbf{1}_0+\frac{\varkappa-1}{2\varkappa}a_{-2}\mathbf{1}_0+\frac{1}{\varkappa}\,\mathbf{1}_{-2},z\right),$$

and similarly for (3.3), which means that the vertex algebra structure is compatible with $\mathrm{Vir}_c\oplus\mathrm{Vir}_{\bar c}$-module structure on $\tilde{\mathbb{F}}_\varkappa$.

We introduce the VOA structure in $\tilde{\mathbb{F}}_\varkappa$ by setting $\mathcal{L}(z)=L(z)+\bar L(z)=L^F(z)+\bar L^F(z)$. One immediately checks that $\mathcal{L}(z)$ is a Virasoro quantum field with central charge 26, and satisfies $\mathcal{L}_{-1}\mathbf{1}_0=0$. It suffices to check the remaining relation

$$[\mathcal{L}_{-1},\mathcal{Y}(v,z)]=\frac{d}{dz}\mathcal{Y}(v,z),\quad v\in\tilde{\mathbb{F}}_\varkappa, \tag{3.4}$$

for each of the generating quantum fields, which is done by direct computations. □

3.3. $\mathrm{Vir}_c\oplus\mathrm{Vir}_{\bar c}$**-module structure of $\tilde{\mathbb{F}}_\varkappa$ for generic $\varkappa$.** We now describe the socle filtration of the $\mathrm{Vir}_c\oplus\mathrm{Vir}_{\bar c}$-module $\tilde{\mathbb{F}}_\varkappa$ for generic $\varkappa$, when it is completely analogous to the finite-dimensional and affine cases, given by Theorem 1.3 and Theorem 2.6. In this subsection we assume that

$$\varkappa\notin\mathbb{Q},\qquad c=13-\frac{6}{\varkappa}-6\varkappa,\qquad \bar c=13+\frac{6}{\varkappa}+6\varkappa.$$

For a Vir_c-module $\tilde V$, the restricted dual space $\tilde V'$ can be equipped with a Vir_c-action by

$$\langle L_n\,v',v\rangle=\langle v',L_{-n}v\rangle.$$

We denote the resulting dual module by $\tilde V^\star$.

We denote by $\tilde V_{\Delta,c}$ the irreducible Vir_c-module, generated by a highest weight vector $\tilde v$ satisfying $L_0\,\tilde v=\Delta\,\tilde v$ and $L_n\tilde v=0$ for $n>0$. For any $\lambda\in\mathfrak{h}^*$, set

$$\Delta(\lambda)=\frac{\lambda(\lambda+2)}{4\varkappa}-\frac{\lambda}{2},\qquad \bar\Delta(\lambda)=-\frac{\lambda(\lambda+2)}{4\varkappa}-\frac{\lambda}{2}.$$

Theorem 3.4. *There exists a filtration*

$$0\subset\tilde{\mathbb{F}}^{(0)}_\varkappa\subset\tilde{\mathbb{F}}^{(1)}_\varkappa\subset\tilde{\mathbb{F}}^{(2)}_\varkappa=\tilde{\mathbb{F}}_\varkappa \tag{3.5}$$

of $\mathrm{Vir}_c \oplus \mathrm{Vir}_{\bar{c}}$*-submodules of* $\tilde{\mathbb{F}}_\varkappa$ *such that*

$$\tilde{\mathbb{F}}^{(2)}_\varkappa / \tilde{\mathbb{F}}^{(1)}_\varkappa \cong \bigoplus_{\lambda \in \mathbf{P}^+} \tilde{V}_{\Delta(-\lambda-2),c} \otimes \tilde{V}^\star_{\bar{\Delta}(-\lambda-2),\bar{c}}, \tag{3.6}$$

$$\tilde{\mathbb{F}}^{(1)}_\varkappa / \tilde{\mathbb{F}}^{(0)}_\varkappa \cong \bigoplus_{\lambda \in \mathbf{P}^+} \left(\tilde{V}_{\Delta(\lambda),c} \otimes \tilde{V}^\star_{\bar{\Delta}(-\lambda-2),\bar{c}} \oplus \tilde{V}_{\Delta(-\lambda-2),c} \otimes \tilde{V}^\star_{\bar{\Delta}(\lambda),\bar{c}} \right), \tag{3.7}$$

$$\tilde{\mathbb{F}}^{(0)}_\varkappa \cong \bigoplus_{\lambda \in \mathbf{P}} \tilde{V}_{\Delta(\lambda),c} \otimes \tilde{V}^\star_{\bar{\Delta}(\lambda),\bar{c}}. \tag{3.8}$$

Proof. One can derive from Proposition 3.2 that the for generic $\varkappa$ the reduction sends exact sequences of $\hat{\mathfrak{g}}_k$-modules to exact sequences of Vir_c-modules, which implies in particular that

$$H^n_{DS}(\hat{V}_{\lambda,k}) = \begin{cases} \tilde{V}_{\Delta(\lambda),c}, & n = 0 \\ 0, & n \neq 0 \end{cases},$$

It is then easy to check that the images $\tilde{\mathbb{F}}^{(0,1,2)}_\varkappa$ of the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodules $\hat{\mathbb{F}}^{(0,1,2)}_\varkappa$ from Theorem 2.6 under the two-sided quantum Drinfeld-Sokolov reduction satisfy the required properties. □

An alternative direct approach repeats the steps in the proof of Theorem 1.3. In particular, we get a decomposition into blocks,

$$\tilde{\mathbb{F}}_\varkappa = \tilde{\mathbb{F}}_\varkappa(-1) \oplus \bigoplus_{\lambda \in \mathbf{P}^+} \tilde{\mathbb{F}}_\varkappa(\lambda). \tag{3.9}$$

We also have the following Virasoro analogue of Corollary 1.4 and Theorem 2.7.

Theorem 3.5. *There exists a subspace* $\tilde{\mathbf{F}}_\varkappa \subset \tilde{\mathbb{F}}_\varkappa$*, satisfying*

1. $\tilde{\mathbf{F}}_\varkappa$ *is a vertex operator subalgebra of* $\tilde{\mathbb{F}}_\varkappa$*, and is generated by the quantum fields* (2.18), (2.19) *and* $\mathbb{Y}(1, z)$*. In particular,* $\tilde{\mathbf{F}}_\varkappa$ *is a* $\mathrm{Vir}_c \oplus \mathrm{Vir}_{\bar{c}}$*-submodule of* $\tilde{\mathbb{F}}_\varkappa$*.*
2. *As a* $\mathrm{Vir}_c \oplus \mathrm{Vir}_{\bar{c}}$*-module,* $\tilde{\mathbf{F}}_\varkappa$ *is generated by the vectors* $\{\mathbf{1}_\lambda\}_{\lambda \in \mathbf{P}^+}$*, and we have*

$$\tilde{\mathbf{F}}_\varkappa \cong \bigoplus_{\lambda \in \mathbf{P}^+} \tilde{V}_{\Delta(\lambda),c} \otimes \tilde{V}^\star_{\bar{\Delta}(\lambda),\bar{c}}. \tag{3.10}$$

Proof. The desired subspace $\tilde{\mathbf{F}}_\varkappa$ is the image of the vertex subalgebra $\hat{\mathbf{F}}_\varkappa$ under the two-sided quantum Drinfeld-Sokolov reduction. We leave technical details to the reader. □

3.4. Semi-infinite cohomology of Vir.

The central charge for the diagonal action of Vir in the modified regular representations is equal to the special value 26. In this section we study the semi-infinite cohomology of Vir with coefficients in $\tilde{\mathbb{F}}_\varkappa$ and in $\tilde{\mathbf{F}}_\varkappa$.

The properties of the appropriate "space of semi-infinite forms" $\tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}}$ for the Virasoro algebra are summarized in the following

Proposition 3.6. *Set* $\tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}} = \bigwedge \mathrm{Vir}_- \otimes \bigwedge \mathrm{Vir}'_+$*, where* $\mathrm{Vir}_- = \bigoplus_{n \leq -2} \mathbb{C}L_n$ *and* $\mathrm{Vir}_+ = \bigoplus_{n \geq -1} \mathbb{C}L_n$*. Then*

1. *The Clifford algebra, generated by* $\{b_n, c_n\}_{n \in \mathbb{Z}}$ *with relations*

$$\{b_m, b_n\} = \{c_m, c_n\} = 0, \qquad \{b_m, c_n\} = \delta_{m+n,0}. \tag{3.11}$$

acts irreducibly on $\tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}}$, *so that for any* $\omega_- \in \bigwedge \mathrm{Vir}_-$, $\omega_+ \in \bigwedge \mathrm{Vir}'_+$ *we have*

$$b_n(1\otimes\omega) = \begin{cases} 0, & n \geq -1 \\ 1\otimes(L_n\wedge\omega), & n\leq -2 \end{cases}, \qquad c_n(\omega\otimes 1) \quad = \begin{cases} (L'_{-n}\wedge\omega)\otimes 1, & n\leq 1 \\ 0, & n\geq 2 \end{cases}.$$

(2) $\tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}}$ *is a bi-graded vertex superalgebra, with vacuum* $\mathbf{1} = 1\otimes 1$, *generated by*

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}, \qquad |b(z)| = -1, \quad \deg b(z) = 2,$$
$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}, \qquad |c(z)| = 1, \quad \deg c(z) = -1.$$

(3) $\tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}}$ *has a* Vir*-module structure with central charge* $c = -26$, *defined by*

$$\pi(L_n) = \sum_{m\in\mathbb{Z}} (m-n) : c_{-m} b_{n+m} : .$$

Definition 5. The BRST complex, associated with a Vir-module $\tilde{V}$ with central charge $c = 26$, is the complex $C^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{V}) = \tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}+\bullet}\otimes\tilde{V}$, with the differential

$$\tilde{\mathbf{d}} = \sum_{n\in\mathbb{Z}} c_{-n}\pi_{\tilde{V}}(L_n) - \frac{1}{2}\sum_{m,n\in\mathbb{Z}} (m-n) : c_{-m}c_{-n}b_{m+n} : . \tag{3.12}$$

The corresponding cohomology is denoted $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{V})$.

As in the affine case, the special value $c = 26$ of the central charge is required to ensure that $\tilde{\mathbf{d}}^2 = 0$.

Theorem 3.7. *The vertex superalgebras* $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa)$ *and* $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa)$ *degenerate into the commutative superalgebras, and we have commutative algebra isomorphisms*

$$H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa) \cong H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbf{F}}_\varkappa), \qquad H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa) \cong H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_\varkappa). \tag{3.13}$$

In particular,

$$H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa) \cong H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa) \cong \mathbb{C}[\mathbf{P}]^W.$$

Proof. The problem of computing the semi-infinite cohomology of Vir, as well as its inherited algebra structure, has been extensively studied by mathematicians and physicists working in the string theory. We take advantage of these results, and construct our proof by combining entire blocks from previous papers.

We note that for both $\tilde{\mathbf{F}}_\varkappa$ and $\tilde{\mathbb{F}}_\varkappa$ the diagonal action of Vir does not contain additional vertex operator shifts, and is equal to the sum of two standard Feigin-Fuks actions.

The comprehensive answer for the cohomology of tensor products of Feigin-Fuks and/or irreducible modules was given in [LZ2] for the most difficult case of the central charge $c = c_{p,q}$, corresponding to $\varkappa = \frac{p}{q} \in \mathbb{Q}$. Simplified (for the case of generic $\varkappa$) version of their computations, and the spectral sequence associated with filtrations of Theorem 3.4, yield

$$H^{\frac{\infty}{2}+n}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa(\lambda)) = \begin{cases} \mathbb{C}, & n = 0, 3 \\ 0, & \text{otherwise} \end{cases},$$

$$H^{\frac{\infty}{2}+n}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa(-1)) = \begin{cases} \mathbb{C}, & n = 1, 2 \\ 0, & \text{otherwise} \end{cases}, \qquad H^{\frac{\infty}{2}+n}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa(\lambda)) = \begin{cases} \mathbb{C}, & n = 0, 2 \\ \mathbb{C}^2, & n = 1 \\ 0, & \text{otherwise} \end{cases},$$

for each $\lambda \in \mathbf{P}^+$, as well as natural isomorphisms

$$H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa(\lambda)) \cong H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa(\lambda)).$$

The algebra structure of $H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa)$ is in fact independent of $\varkappa$, as can be seen from the change of variables

$$p_n = \frac{a_n + \bar{a}_n}{2}, \qquad q_n = \frac{a_n - \bar{a}_n}{2\varkappa}.$$

Indeed, the new commutation relations become $[p_m, p_n] = [q_m, q_n] = 0$ and $[p_m, q_n] = \delta_{m+n,0}$, and the diagonal Virasoro action is given by

$$\mathcal{L}(z) =: p(z)q(z) : +p(z) - q(z).$$

For the special case $\varkappa = 1$, corresponding to the pairing of $c = 1$ and $\bar{c} = 25$ modules, the cohomology of a bigger vertex algebra $\mathcal{A}_{2D} = \bigoplus_{\lambda,\mu\in\mathbb{Z}} \tilde{F}_{\lambda,1} \otimes \tilde{F}_{\mu,-1}$ was identified in [WZ] with the polynomial algebra $\mathbb{C}[x, y]$ in two variables. The subalgebra $H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}c; \tilde{\mathbb{F}}_\varkappa)$ is therefore isomorphic to the polynomial algebra $\mathbb{C}[\chi]$, and we can take any nonzero cohomology class $\chi \in H^{\frac{\infty}{2}+0}(\mathrm{Vir}, \mathbb{C}c; \tilde{\mathbb{F}}_\varkappa(1))$ as the generator.

The vertex superalgebra structures on $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa)$ and $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa)$ degenerate into commutative superalgebras. It is clear that both are free $\mathbb{C}[\chi]$-modules.

It follows immediately that $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa) \cong \mathbb{C}[\chi] \otimes \bigwedge^\bullet[\eta]$, where we can pick any non-zero element $\eta \in H^{\frac{\infty}{2}+3}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa(0))$. This settles the case of $\tilde{\mathbf{F}}_\varkappa$.

To get the generators of $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa)$, we pick non-zero representatives

$$\xi_{-1} \in H^{\frac{\infty}{2}+1}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa(-1)), \qquad \eta_0 \in H^{\frac{\infty}{2}+1}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa(0)),$$

such that η_0 is not proportional to $\chi \cdot \xi_{-1}$. One can check that $\eta_0\xi_{-1} \neq 0$, and as in Theorem 1.12 it follows that $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa) \cong \mathbb{C}[\chi] \otimes \bigwedge^\bullet[\xi_{-1}, \eta_0]$. The statement now follows from Theorem 1.12 and Corollary 2.10. □

Remark 7. It would be nice to get a direct proof of isomorphisms (3.13) by using the techniques of the quantum Drinfeld-Sokolov reduction.

4. Extensions, generalizations, conjectures

4.1. **Heterogeneous vertex operator algebra.** As we mentioned above, the vertex algebra construction for the Virasoro algebra can be obtained from their affine analogues by applying the two-sided quantum Drinfeld-Sokolov reduction to the left and right $\hat{\mathfrak{g}}$-action. One can consider a similar construction where the reduction is only applied to the affine action on one side, thus leading to a vertex operator algebra with two commuting actions of $\hat{\mathfrak{g}}_k$ and $\mathrm{Vir}_{\bar{c}}$ with appropriate $k, \bar{c}$. Indeed, one can see that the following gives a direct realization of such a vertex algebra.

Theorem 4.1. *Let $\varkappa \neq 0$. Set $k = \varkappa - h^\vee$, $\bar{c} = 13 + 6\varkappa + \frac{6}{\varkappa}$, and $\check{\mathbb{F}}_\varkappa = \hat{F}(\beta, \gamma) \otimes \tilde{\mathbb{F}}_\varkappa$. Then*

(1) *The space* $\check{\mathbb{F}}_\varkappa$ *has a* $\hat{\mathfrak{g}}_k \oplus \mathrm{Vir}_{\bar{c}}$*-module structure, defined by*

$$\begin{aligned}
\mathbf{e}(z) &= \gamma(z),\\
\mathbf{h}(z) &= 2 : \beta(z)\gamma(z) : + a(z),\\
\mathbf{f}(z) &= - : \beta(z)^2\gamma(z) : - \beta(z)a(z) - k\beta'(z) - \mathbb{Y}(-2, z),
\end{aligned} \tag{4.1}$$

$$\bar{L}(z) = -\frac{: \bar{a}(z)^2 :}{4\varkappa} + \frac{\varkappa+1}{2\varkappa}\bar{a}'(z) + \frac{1}{\varkappa}\mathbb{Y}(-2, z)\gamma(z). \tag{4.2}$$

(2) *The space* $\check{\mathbb{F}}_\varkappa$ *has a compatible VOA structure with* $\operatorname{rank}\check{\mathbb{F}}_\varkappa = 28$.

Proof. The verification of commutation relations is straightforward. We define the Virasoro quantum field by

$$\mathcal{L}(z) = \frac{1}{2\varkappa}\left(\frac{: \mathbf{h}(z)^2 :}{2} + : \mathbf{e}(z)\mathbf{f}(z) : + : \mathbf{f}(z)\mathbf{e}(z) :\right) + \frac{\mathbf{h}'(z)}{2} + \bar{L}(z) \tag{4.3}$$

The central charge for the Sugawara construction modified by $\frac{\mathbf{h}'(z)}{2}$ is equal to $\frac{3k}{k+h^\vee} - 6k$, and we compute

$$\operatorname{rank}\check{\mathbb{F}}_\varkappa = \left(\frac{3(\varkappa - h^\vee)}{\varkappa} - 6(\varkappa - h^\vee)\right) + \left(13 + \frac{6}{\varkappa} + 6\varkappa\right) = 28.$$

□

We will call the vertex operator algebra of Theorem 4.1 the heterogeneous VOA. Note (see [L] and references therein) that the central charge $c = 28$ appears as the critical value in the study of 2D gravity in the light-cone gauge!

The structure of the bimodule $\check{\mathbb{F}}_\varkappa$ in the generic case is again quite similar to the non-semisimple bimodule $\mathfrak{R}(G_0)$. From now on we assume that

$$\varkappa \notin \mathbb{Q}, \qquad k = \varkappa - h^\vee, \qquad \bar{c} = 13 + \frac{6}{\varkappa} + 6\varkappa.$$

Theorem 4.2. *There exists a filtration*

$$0 \subset \check{\mathbb{F}}_\varkappa^{(0)} \subset \check{\mathbb{F}}_\varkappa^{(1)} \subset \check{\mathbb{F}}_\varkappa^{(2)} = \check{\mathbb{F}}_\varkappa$$

of $\hat{\mathfrak{g}}_k \oplus \mathrm{Vir}_{\bar{c}}$*-submodules of* $\check{\mathbb{F}}_\varkappa$*, such that*

$$\check{\mathbb{F}}_\varkappa^{(2)}/\check{\mathbb{F}}_\varkappa^{(1)} \cong \bigoplus_{\lambda\in\mathbf{P}^+} \hat{V}_{-\lambda-2,k} \otimes \tilde{V}^\star_{\bar{\Delta}(-\lambda-2),\bar{c}}, \tag{4.4}$$

$$\check{\mathbb{F}}_\varkappa^{(1)}/\check{\mathbb{F}}_\varkappa^{(0)} \cong \bigoplus_{\lambda\in\mathbf{P}^+} \left(\hat{V}_{\lambda,k} \otimes \tilde{V}^\star_{\bar{\Delta}(-\lambda-2),\bar{c}} \oplus \hat{V}_{-\lambda-2,k} \otimes \tilde{V}^\star_{\bar{\Delta}(\lambda),\bar{c}}\right), \tag{4.5}$$

$$\check{\mathbb{F}}_\varkappa^{(0)} \cong \bigoplus_{\lambda\in\mathbf{P}} \hat{V}_{\lambda,k} \otimes \tilde{V}^\star_{\bar{\Delta}(\lambda),\bar{c}}. \tag{4.6}$$

The heterogeneous VOA contains a vertex operator subalgebra, analogous to the classical Peter-Weyl subalgebra $\mathfrak{R}(G) \subset \mathfrak{R}(G_0)$.

Theorem 4.3. *There exists a subspace* $\check{\mathbf{F}}_\varkappa \subset \check{\mathbb{F}}_\varkappa$*, satisfying*

(1) $\check{\mathbf{F}}_\varkappa$ *is a vertex operator subalgebra of* $\check{\mathbb{F}}_\varkappa$*, and is generated by the fields* (4.1), (4.2), *and* $\mathbb{Y}(1, z)$*. In particular,* $\check{\mathbf{F}}_\varkappa$ *is a* $\hat{\mathfrak{g}}_k \oplus \mathrm{Vir}_{\bar{c}}$*-submodule of* $\check{\mathbb{F}}_\varkappa$*.*

(2) *As a* $\hat{\mathfrak{g}}_k \oplus \mathrm{Vir}_{\bar{c}}$*-module,* $\check{\mathbf{F}}_\varkappa$ *is generated by the vectors* $\{\mathbf{1}_\lambda\}_{\lambda\in\mathbf{P}^+}$*, and we have*

$$\check{\mathbf{F}}_\varkappa \cong \bigoplus_{\lambda\in\mathbf{P}^+} \hat{V}_{\lambda,k} \otimes \tilde{V}_{\bar{\Delta}(\lambda),\bar{c}}. \tag{4.7}$$

The proofs of the above theorems are obtained from their affine counterparts by applying the quantum Drinfeld-Sokolov reduction to (right) $\hat{\mathfrak{g}}_{\bar{k}}$-action, similarly to the Virasoro case.

The fact that the ranks of VOAs $\check{\mathbb{F}}_\varkappa$ and $\check{\mathbf{F}}_\varkappa$ are equal to 28 naturally leads to the consideration of the semi-infinite cohomology of these modules. We note that although the total Virasoro quantum field $\mathcal{L}(z)$ does not commute with $\hat{\mathfrak{g}}$, the spaces $\check{\mathbb{F}}_\varkappa$ and $\check{\mathbf{F}}_\varkappa$ can be regarded as modules over the semi-direct product $\mathrm{Vir} \ltimes \hat{\mathfrak{g}}$, such that

$$\begin{aligned}
[\mathcal{L}_m, \mathbf{e}_n] &= -(n+m+1)\, \mathbf{e}_{m+n}, \\
[\mathcal{L}_m, \mathbf{f}_n] &= (m-n+1)\, \mathbf{f}_{m+n}, \\
[\mathcal{L}_m, \mathbf{h}_n] &= -n\, \mathbf{h}_{m+n} + m(m+1)\delta_{m+n,0}\, k.
\end{aligned} \tag{4.8}$$

The semi-infinite cohomology is defined by the BRST complex of the subalgebra $\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+$, where as before $\hat{\mathfrak{n}}_+ = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\mathbf{e}_n$ is the nilpotent loop subalgebra of $\hat{\mathfrak{g}}$. The condition $c = 28$ ensures that the differential squares to zero.

Definition 6. The BRST complex, associated with a $\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+$-module $\check{V}$ with Virasoro central charge $c = 28$ is the complex $C^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \check{V}) = \tilde{\mathbf{\Lambda}}^{\frac{\infty}{2}+\bullet} \otimes \hat{\Lambda}(\psi, \psi^*) \otimes \check{V}$, with the differential

$$\begin{aligned}
\check{\mathbf{d}} = &\sum_{n\in\mathbb{Z}} c_{-n}\pi_{\check{V}}(\mathcal{L}_n) - \frac{1}{2}\sum_{m,n\in\mathbb{Z}} (m-n) :c_{-m}c_{-n}b_{m+n}: + \\
&+ \sum_{n\in\mathbb{Z}} \psi^*_{-n}\pi_{\check{V}}(\mathbf{e}_n) - \sum_{m,n\in\mathbb{Z}} -(m+n+1)c_{-m} :\psi^*_{-n}\psi_{m+n}:
\end{aligned} \tag{4.9}$$

The corresponding cohomology is denoted $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{V})$.

Proposition 4.4. *The vertex algebras* $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{\mathbf{F}}_\varkappa)$ *and* $H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}; \check{\mathbf{F}}_\varkappa)$ *degenerate into commutative algebra structures, and we have isomorphisms*

$$H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{\mathbf{F}}_\varkappa) \cong H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbf{F}}_\varkappa),$$

$$H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{\mathbb{F}}_\varkappa) \cong H^{\frac{\infty}{2}+\bullet}(\mathrm{Vir}, \mathbb{C}\mathbf{c}; \tilde{\mathbb{F}}_\varkappa).$$

In particular,

$$H^{\frac{\infty}{2}+0}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{\mathbf{F}}_\varkappa) \cong H^{\frac{\infty}{2}+0}(\mathrm{Vir} \ltimes \hat{\mathfrak{n}}_+, \mathbb{C}\mathbf{c}; \check{\mathbb{F}}_\varkappa) \cong \mathbb{C}[\mathbf{P}]^W.$$

Proof. We use the technique from [L], where similar isomorphisms were established for relative cohomology spaces. Let $\mathbf{d}_{\mathfrak{n}_+} = \sum_{n\in\mathbb{Z}} \psi^*_{-n}\pi(\mathbf{e}_n)$ be the BRST differential for $\mathfrak{n}_+$; one can show that $\mathbf{d}^2_{\mathfrak{n}_+} = 0 = \{\mathbf{d}_{\mathfrak{n}_+}, \check{\mathbf{d}}\}$, which leads to the spectral sequence associated with decomposition $\check{\mathbf{d}} = \mathbf{d}_{\mathfrak{n}_+} + (\check{\mathbf{d}} - \mathbf{d}_{\mathfrak{n}_+})$. Computing the cohomology with respect to $\mathbf{d}_{\mathfrak{n}_+}$ first, and using Proposition 3.2, we get the desired statement. For full technical details (there is a slight difference between BRST reduction for $\mathfrak{n}_+$ and quantum Drinfeld-Sokolov reduction, but it doesn't affect the outcome) we refer the reader to [L]. □

4.2. **General construction of vertex operator algebras and equivalence of categories.** The vertex operator algebras constructed in the previous sections can be built, like conformal field theories, by pairing the left and right modules from certain equivalent categories of representations of infinite-dimensional Lie algebras. The operators $\mathcal{Y}(\cdot, z)$ are then constructed by pairing the left and right intertwining operators. There is a unique choice of the structural coefficients for such pairing that would ensure the locality condition for the vertex operator algebras; these coefficients are determined by the tensor structure on the category of representations. Conversely, a natural VOA structure on a bimodule can be used to establish the equivalence of the left and right tensor categories.

The vertex operator algebra constructions in this paper deal with the pairings of different categories of modules. In the affine case, we pair the $\hat{\mathfrak{g}}$ modules on levels k and $\bar{k} = -2h^\vee - k$, symmetric with respect to the critical level $-h^\vee$; the equivalence of the corresponding tensor categories was studied in [Fi]. In the Virasoro case, we pair the modules with central charges c and $\bar{c} = 26 - c$.

The theorems of Peter-Weyl type can be extended to the quantum group $\mathcal{U}_q(\mathfrak{g})$, associated with G. On one hand, the modules from the category $\mathcal{O}$ can be q-deformed into modules over $\mathcal{U}_q(\mathfrak{g})$; on the other hand, one can define q-deformations $\mathfrak{R}_q(G)$ and $\mathfrak{R}_q(G_0)$ of the algebras of regular functions, which have especially simple description for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. When q is not a root of unity, we have the quantum analogues of isomorphisms (0.1), (0.5).

The Drinfeld-Kohno theorem establishes an isomorphism of tensor categories of representations of the quantum groups and affine Lie algebras when $q = \exp\left(\frac{\pi i}{k+h^\vee}\right)$. This equivalence, also extended to $\mathcal{W}$-algebras, was made explicit in [S], where intertwining operators for $\mathcal{U}_q(\mathfrak{g})$ were directly identified with their VOA counterparts for $\hat{\mathfrak{g}}_k$ and $\mathcal{W}(\hat{\mathfrak{g}}_k)$; the key ingredients were the geometric results in [V] on the homology of configuration spaces. The construction in [S] built conformal field theories, associated to affine Lie algebras and $\mathcal{W}$-algebras based on their quantum group counterparts, and can be modified to produce the vertex algebras discussed in this paper.

The Drinfeld-Kohno equivalence also allows to couple categories of different types, producing in particular the heterogeneous VOA of the previous subsection. Another important case is the Frenkel-Kac construction, which corresponds to the pairing of modules for $\mathfrak{g}$ and $\mathcal{W}(\hat{\mathfrak{g}})$ with central charge $c = \dim \mathfrak{h}$ (see [F]). However, in general pairings between the quantum group and the affine Lie (or $\mathcal{W}$-) algebra lead to the generalized vertex algebra structures, satisfying a braided version of the commutativity axiom. An example of such structure was proposed in [MR].

4.3. **Integral central charge, semi-infinite cohomology and Verlinde algebras.** In this work we studied the structure of the generalized Peter-Weyl bimodules for $\hat{\mathfrak{g}}$ only for the generic values $k \notin \mathbb{Q}$ of the central charge. The structure of these bimodules when k is integral is more complex and undoubtedly even more interesting. In the most special case when $k = \bar{k} = -h^\vee$, we get a regular representation of the affine Lie algebra $\hat{\mathfrak{g}}$ at the critical level, which can be viewed as the direct counterpart of the finite-dimensional case. This space admits a realization as a certain space of meromorphic functions on the affine Lie group $\hat{G}$, and subspaces of spherical functions with respect to conjugation give rise to solutions of the quantum elliptic Calogero-Sutherland system, generalizing the trigonometric analogue in the finite-dimensional case [EFK].

Another special case is $k = -h^\vee + 1, \bar{k} = -h^\vee - 1$, when the quantum group degenerates into its classical counterpart. In this case the left and right Fock spaces used in our construction

each have separate vertex algebra structures, and the operators $\mathbb{Y}(-2,w)$, which play an important role in this paper, are factored into products of left and right vertex operators used in the basic representations of $\hat{\mathfrak{g}}$. The semi-infinite cohomology of the corresponding $\mathcal{W}$-algebras is fundamental to the string theory, and was studied in [WZ] for the Virasoro algebra, and in [BMP] for $\mathcal{W}_3$.

For positive integral k one expects the existence of truncated versions $\hat{\mathbf{F}}_{k+h^\vee}$ and $\hat{\mathbb{F}}_{k+h^\vee}$ of our vertex operators algebras, similar to the truncation in the conformal field theory, where the positive dominant cone $\mathbf{P}^+$ is replaced by the alcove $\mathbf{P}^+_k \subset \mathbf{P}^+$. Then the relative semi-infinite cohomology of $\hat{\mathfrak{g}}$ with coefficients in $\hat{\mathbf{F}}_{k+h^\vee}$ and $\hat{\mathbb{F}}_{k+h^\vee}$ with respect to the center should be truncated correspondingly. The identification of the zero semi-infinite cohomology groups with the representation ring of G in Corollary 2.10 leads to the following conjecture.

Conjecture 1. *For positive integral k, let $\mathbf{V}_k(\hat{\mathfrak{g}})$ denote the Verlinde algebra associated with integrable level k representations of $\hat{\mathfrak{g}}$, and let $\mathbb{V}_k(\hat{\mathfrak{g}})$ denote its counterpart associated to the big projective modules (see* [La]*). Then we have commutative algebra isomorphisms*

$$H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbf{F}}_{k+h^\vee}) \cong \mathbf{V}_k(\hat{\mathfrak{g}}), \qquad H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_{k+h^\vee}) \cong \mathbb{V}_k(\hat{\mathfrak{g}}).$$

In other words, the most essential part of the VOA structure, embodied in the 0th cohomology, is equivalent to the structure of the fusion rules of the tensor category of $\hat{\mathfrak{g}}$-modules, encoded in the Verlinde algebra.

It was also realized recently (see [FHT] and references therein) that the Verlinde algebra $\mathbf{V}_k(\hat{\mathfrak{g}})$ admits an alternative realization in terms of twisted equivariant K-theory ${}^{k+h^\vee}K_G^{\dim G}(G)$ of a compact simple Lie group G (which, in the notations of [FHT], is the compact form of the complex Lie group which we denoted by G in this paper).

Thanks to the results of [FHT], the first isomorphism of our conjecture can be restated in a more invariant form.

Conjecture 2. *We have a natural commutative algebra isomorphism*

$$H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \mathfrak{R}'_{k+h^\vee}(\hat{G})) \cong {}^{k+h^\vee}K_G^{\dim G}(G).$$

A conceivable direct geometric proof of the last isomorphism might combine the realization of the left hand side using the works [GMS] and [AG] with the interpretation of the right hand side given in the works [FHT] and [AS]. A similar K-theoretic interpretation of $H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}, \mathbb{C}\mathbf{k}; \hat{\mathbb{F}}_{k+h^\vee})$ in our second conjecture might add another twirl to the twisted equivariant K-theory.

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